

# STATISTICAL PROPERTIES OF UNIMODAL MAPS: THE QUADRATIC FAMILY

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**ABSTRACT.** We prove that almost every non-regular real quadratic map is Collet-Eckmann and has polynomial recurrence of the critical orbit (proving a conjecture by Sinai). It follows that typical quadratic maps have excellent ergodic properties, as exponential decay of correlations (Keller and Nowicki, Young) and stochastic stability in the strong sense (Baladi and Viana). This is an important step to get the same results for more general families of unimodal maps.

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## INTRODUCTION

Here we consider the quadratic family,  $f_a = a - x^2$ , where  $-1/4 \leq a \leq 2$  is the parameter, and we analyze its dynamics in the invariant interval.

The quadratic family has been one of the most studied dynamical systems in the last decades. It is one of the most basic examples and exhibits a very rich behavior. It was also studied through many different techniques. Here we are interested in describing the dynamics of a typical quadratic map from the statistical point of view.

**0.1. The probabilistic point of view in dynamics.** In the last decade Palis [Pa] described a general program for (dissipative) dynamical systems in any dimension. In short, he asks to show that ‘typical’ dynamical systems can be modeled stochastically in a robust way. More precisely, one should show that such typical

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system can be described by finitely many attractors, each of them supporting a (ergodic) physical measure: time averages of Lebesgue almost every orbit should converge to spatial averages according to one of the physical measures. The description should be robust under (sufficiently) random perturbations of the system: one asks for stochastic stability.

Moreover, a typical dynamical system was to be understood in the Kolmogorov sense: as a set of full measure in generic parametrized families.

Besides the questions posed by this conjecture, much more can be asked about the statistical description of the long term behavior of a typical system. For instance, the definition of physical measure is related to the validity of the Law of Large Numbers. Are other theorems still valid, like the Central Limit or Large Deviation theorems? Those questions are usually related to the rates of mixing of the physical measure.

**0.2. The richness of the quadratic family.** While we seem still very far away of any description of dynamics of typical dynamical systems (even in one-dimension), the quadratic family has been a remarkable exception. Let us describe briefly some results which show the richness of the quadratic family from the probabilistic point of view.

The initial step in this direction was the work of Jakobson [J], where it was shown that for a positive measure set of parameters the behavior is stochastic, more precisely, there is an absolutely continuous invariant measure (the physical measure) with positive Lyapunov exponent: for Lebesgue almost every  $x$ ,  $|Df^n(x)|$  grows exponentially fast. On the other hand, it was later shown by Lyubich [L2] and Graczyk-Swiatek [GS1] that regular parameters (with a periodic hyperbolic attractor) are (open and) dense. While stochastic parameters are predominantly expanding (in particular have sensitive dependence to initial conditions), regular parameters are deterministic (given by the periodic attractor). So at least two kinds of very distinct observable behavior are present in the quadratic family, and they alternate in a complicate way.

It was later shown that stochastic behavior could be concluded from enough expansion along the orbit of the critical value: the Collet-Eckmann condition, exponential growth of  $|Df^n(f(0))|$ , was enough to conclude a positive Lyapunov exponent of the system. A different approach to Jakobson's Theorem in [BC1] and [BC2] focused specifically on this property: the set of Collet-Eckmann maps has positive measure. After these initial works, many others studied such parameters (sometimes with extra assumptions), obtaining refined information of the dynamics of CE maps, particularly information about exponential decay of correlations<sup>1</sup> (Keller and Nowicki in [KN] and Young in [Y]), and stochastic stability (Baladi and Viana in [BV]). The dynamical systems considered in those papers have generally been shown to have excellent statistical description<sup>2</sup>.

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<sup>1</sup>CE quadratic maps are not always mixing and finite periodicity can appear in a robust way. This phenomena is related to the map being renormalizable, and this is the only obstruction: the system is exponentially mixing after renormalization.

<sup>2</sup>It is now known that weaker expansion than Collet-Eckmann is enough to obtain stochastic behavior for quadratic maps, on the other hand, exponential decay of correlations is actually equivalent to the CE condition [NS], and all current results on stochastic stability use the Collet-Eckmann condition.

Many of those results also generalized to more general families and sometimes to higher dimensions, as in the case of Hénon maps [BC2].

The main motivation behind this strong effort to understand the class of CE maps was certainly the fact that such a class was known to have positive measure. It was known however that very different (sometimes wild) behavior coexisted. For instance, it was shown the existence of quadratic maps without a physical measure or quadratic maps with a physical measure concentrated on a *repelling* hyperbolic fixed point ([Jo], [HK]). It remained to see if wild behavior was observable.

In a big project in the last decade, Lyubich [L3] together with Martens and Nowicki [MN] showed that almost all parameters have physical measures: more precisely, besides regular and stochastic behavior, only one more behavior could (possibly) happen with positive measure, namely infinitely renormalizable maps (which always have a uniquely ergodic physical measure). Later Lyubich in [L5] showed that infinitely renormalizable parameters have measure zero, thus establishing the celebrated *regular or stochastic* dichotomy. This further advancement in the comprehension of the nature of the statistical behavior of typical quadratic maps is remarkably linked to the progress obtained by Lyubich on the answer of the Feigenbaum conjectures [L4].

**0.3. Statements of the results.** In this work we describe the asymptotic behavior of the critical orbit. Our first result is an estimate of hyperbolicity:

**Theorem A.** *Almost every non-regular real quadratic map satisfies the Collet-Eckmann condition:*

$$\liminf_{n \rightarrow \infty} \frac{\ln(|Df^n(f(0))|)}{n} > 0.$$

The second is an estimate on the recurrence of the critical point. For regular maps, the critical point is non-recurrent (it actually converges to the periodic attractor). Among non-regular maps, however, the recurrence occurs at a precise rate which we estimate:

**Theorem B.** *Almost every non-regular real quadratic map has polynomial recurrence of the critical orbit with exponent 1:*

$$\limsup_{n \rightarrow \infty} \frac{-\ln(|f^n(0)|)}{\ln(n)} = 1.$$

*In other words, the set of  $n$  such that  $|f^n(0)| < n^{-\gamma}$  is finite if  $\gamma > 1$  and infinite if  $\gamma < 1$ .*

As far as we know, this is the first proof of polynomial estimates for the recurrence of the critical orbit valid for a positive measure set of non-hyperbolic parameters (although subexponential estimates were known before). This also answers a long standing conjecture of Sinai.

Theorems A and B show that a typical non regular quadratic maps have enough good properties to conclude the results on exponential decay of correlations (which can be used to prove Central Limit and Large Deviation theorems) and stochastic stability in the sense of  $L^1$  convergence of the densities (of stationary measures of perturbed systems). Many other properties also follow, like existence of a spectral gap in [KN] and the recent results on almost sure (stretched exponential) rates of

convergence to equilibrium in [BBM]. In particular, this answers positively Palis conjecture for the quadratic family.

**0.4. Unimodal maps.** Another reason to deal with the quadratic family is that it seems to open the doors to the understanding of unimodal maps. Its universal behavior was first realized in the topological sense, with Milnor-Thurston theory. The Feigenbaum-Coulet-Tresser observations indicated a geometric universality [L4].

A first result in the understanding of measure-theoretical universality was the work of Avila, Lyubich and de Melo [ALM], where it was shown how to relate metrically the parameter spaces of non-trivial analytic families of unimodal maps to the parameter space of the quadratic family. This was proposed as a method to relate observable dynamics in the quadratic family to observable dynamics of general analytic families of unimodal maps. In that work the method is used successfully to extend the regular or stochastic dichotomy to this broader context.

We are also able to adapt those methods to our setting. The techniques developed here and the methods of [ALM] are the main tools used in [AM1] to obtain the main results of this paper (except the exact value of the polynomial recurrence) for non-trivial real analytic families of unimodal maps (with negative Schwarzian derivative and quadratic critical point). This is a rather general set of families, as trivial families form a set of infinite codimension. For a different approach (still based on [ALM]) which does not use negative Schwarzian derivative and obtains the exponent 1 for the polynomial recurrence, see [A].

In [AM1] we also prove a version of Palis conjecture in the smooth setting. There is a residual set of  $k$ -parameter  $C^3$  (for the equivalent  $C^2$  result, see [A]) families of unimodal maps with negative Schwarzian derivative such that almost every parameter is either regular or Collet-Eckmann with subexponential bounds for the recurrence of the critical point.

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## 1. GENERAL DEFINITIONS

**1.1. Maps of the interval.** Let  $f : I \rightarrow I$  be a  $C^1$  map defined on some interval  $I \subset \mathbb{R}$ . The *orbit* of a point  $p \in I$  is the sequence  $\{f^k(p)\}_{k=0}^\infty$ . We say that  $p$  is *recurrent* if there exists a subsequence  $n_k \rightarrow \infty$  such that  $\lim f^{n_k}(p) = p$ .

We say that  $p$  is a *periodic point of period  $n$*  of  $f$  if  $f^n(p) = p$ , and  $n \geq 1$  is minimal with this property. In this case we say that  $p$  is *hyperbolic* if  $|Df^n(p)|$  is not 0 or 1. Hyperbolic periodic orbits are *attracting* or *repelling* according to  $|Df^n(p)| < 1$  or  $|Df^n(p)| > 1$ .

We will often consider the restriction of iterates  $f^n$  to intervals  $T \subset I$ , such that  $f^n|_T$  is a diffeomorphism. In this case we will be interested on the *distortion* of  $f^n|_T$ ,

$$\text{dist}(f^n|_T) = \frac{\sup_T |Df^n|}{\inf_T |Df^n|}.$$

This is always a number bigger than or equal to 1, we will say that it is small if it is close to 1.

**1.2. Trees.** We let  $\Omega$  denote the set of finite sequences of non-zero integers (including the empty sequence). Let  $\Omega_0$  denote  $\Omega$  without the empty sequence. For  $\underline{d} \in \Omega$ ,  $\underline{d} = (j_1, \dots, j_m)$ , we let  $|\underline{d}| = m$  denote its length.

We denote  $\sigma^+ : \Omega_0 \rightarrow \Omega$  by  $\sigma^+(j_1, \dots, j_m) = (j_1, \dots, j_{m-1})$  and  $\sigma^- : \Omega_0 \rightarrow \Omega$  by  $\sigma^-(j_1, \dots, j_m) = (j_2, \dots, j_m)$ .

For the purposes of this paper, one should view  $\Omega$  as a (directed) tree with root  $\underline{d} = \emptyset$  and edges connecting  $\sigma^+(\underline{d})$  to  $\underline{d}$  for each  $\underline{d} \in \Omega_0$ . We will use  $\Omega$  to label objects which are organized in a similar tree structure (for instance, certain families of intervals ordered by inclusion).

**1.3. Growth of functions.** Let  $f : \mathbb{N} \rightarrow \mathbb{R}^+$  be a function. We say that  $f$  grows *at least exponentially* if there exists  $\alpha > 0$  such that  $f(n) > e^{\alpha n}$  for all  $n$  sufficiently big. We say that  $f$  grows *at least polynomially* if there exists  $\alpha > 0$  such that  $f(n) > n^\alpha$  for all  $n$  sufficiently big.

The standard *torrential* function  $T$  is defined recursively by  $T(1) = 1$ ,  $T(n+1) = 2^{T(n)}$ . We say that  $f$  grows *at least torrentially* if there exists  $k > 0$  such that  $f(n) > T(n-k)$  for every  $n$  sufficiently big. We will say that  $f$  grows *torrentially* if there exists  $k > 0$  such that  $T(n-k) < f(n) < T(n+k)$  for every  $n$  sufficiently big.

Torrential growth can be detected from recurrent estimates easily. A sufficient condition for a function  $f$  to grow at least torrentially is an estimate as

$$f(n+1) > e^{f(n)\alpha}$$

for some  $\alpha > 0$ . Torrential growth is implied by an estimate as

$$e^{f(n)\alpha} < f(n+1) < e^{f(n)\beta}$$

with  $0 < \alpha < \beta$ .

We will also say that  $f$  decreases at least exponentially (respectively torrentially) if  $1/f$  grows at least exponentially (respectively torrentially).

**1.4. Quasisymmetric maps.** Let  $k \geq 1$  be given. We say that a homeomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *quasisymmetric* with constant  $k$  if for all  $h > 0$

$$\frac{1}{k} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq k.$$

The space of quasisymmetric maps is a group under composition, and the set of quasisymmetric maps with constant  $k$  preserving a given interval is compact in the uniform topology of compact subsets of  $\mathbb{R}$ . It also follows that quasisymmetric maps are Hölder.

To describe further the properties of quasisymmetric maps, we need the concept of quasiconformal maps and dilatation so we just mention a result of Ahlfors-Beurling which connects both concepts: any quasisymmetric map extends to a quasiconformal real-symmetric map of  $\mathbb{C}$  and, conversely, the restriction of a quasiconformal real-symmetric map of  $\mathbb{C}$  to  $\mathbb{R}$  is quasisymmetric. Furthermore, it is possible to work out upper bounds on the dilatation (of an optimal extension) depending only on  $k$  and conversely.

The constant  $k$  is awkward to work with: the inverse of a quasisymmetric map with constant  $k$  may have a larger constant. We will therefore work with a less

standard constant: we will say that  $h$  is  $\gamma$ -quasisymmetric ( $\gamma$ -qs) if  $h$  admits a quasiconformal symmetric extension to  $\mathbb{C}$  with dilatation bounded by  $\gamma$ . This definition behaves much better: if  $h_1$  is  $\gamma_1$ -qs and  $h_2$  is  $\gamma_2$ -qs then  $h_2 \circ h_1$  is  $\gamma_2 \gamma_1$ -qs.

If  $X \subset \mathbb{R}$  and  $h : X \rightarrow \mathbb{R}$  has a  $\gamma$ -quasisymmetric extension to  $\mathbb{R}$  we will also say that  $h$  is  $\gamma$ -qs.

Let  $QS(\gamma)$  be the set of  $\gamma$ -qs maps of  $\mathbb{R}$ .

## 2. REAL QUADRATIC MAPS

If  $a \in \mathbb{C}$  we let  $f_a : \mathbb{C} \rightarrow \mathbb{C}$  denote the (complex) quadratic map  $a - z^2$ . For real parameters in the range  $-1/4 \leq a \leq 2$ , there exists an interval  $I_a = [\beta, -\beta]$  with

$$\beta = \frac{-1 - \sqrt{1 + 4a}}{2}$$

such that  $f_a(I_a) \subset I_a$  and  $f_a(\partial I_a) \subset \partial I_a$ . For such values of the parameter  $a$ , the map  $f = f_a|_{I_a}$  is unimodal, that is, it is a self map of  $I_a$  with a unique turning point. To simplify the notation, we will usually drop the dependence on the parameter and let  $I = I_a$ .

**2.1. The combinatorics of unimodal maps.** In this subsection we fix a real quadratic map  $f$  and define some objects related to it.

**2.1.1. Return maps.** Given an interval  $T \subset I$  we define the *first return map*  $R_T : X \rightarrow T$  where  $X \subset T$  is the set of points  $x$  such that there exists  $n > 0$  with  $f^n(x) \in T$ , and  $R_T(x) = f^n(x)$  for the minimal  $n$  with this property.

**2.1.2. Nice intervals.** An interval  $T$  is *nice* if it is symmetric around 0 and the iterates of  $\partial T$  never intersect  $\text{int } T$ . Given a nice interval  $T$  we notice that the domain of the first return map  $R_T$  decomposes in a union of intervals  $T^j$ , indexed by integer numbers (if there are only finitely many intervals, some indexes will be corresponded to the empty set). If 0 belongs to the domain of  $R_T$ , we say that  $T$  is *proper*. In this case we reserve the index 0 to denote the component of the critical point:  $0 \in T^0$ .

If  $T$  is nice, it follows that for all  $j \in \mathbb{Z}$ ,  $R_T(\partial T^j) \subset \partial T$ . In particular,  $R_T|_{T^j}$  is a diffeomorphism onto  $T$  unless  $0 \in T^j$  (and in particular  $j = 0$  and  $T$  is proper). If  $T$  is proper,  $R_T|_{T^0}$  is symmetric (even) with a unique critical point 0. As a consequence,  $T^0$  is also a nice interval.

If  $R_T(0) \in T^0$ , we say that  $R_T$  is *central*.

If  $T$  is a proper interval then both  $R_T$  and  $R_{T^0}$  are defined, and we say that  $R_{T^0}$  is the generalized renormalization of  $R_T$ .

**2.1.3. Landing maps.** Given a proper interval  $T$  we define the *landing map*  $L_T : X \rightarrow T^0$  where  $X \subset T$  is the set of points  $x$  such that there exists  $n \geq 0$  with  $f^n(x) \in T^0$ , and  $L_T(x) = f^n(x)$  for the minimal  $n$  with this property. We notice that  $L_T|_{T^0} = \text{id}$ .

**2.1.4. Trees.** We will use  $\Omega$  to label iterations of non-central branches of  $R_T$ , as well as their domains. If  $\underline{d} \in \Omega$ , we define  $T^{\underline{d}}$  inductively in the following way. We let  $T^{\underline{d}} = T$  if  $\underline{d}$  is empty and if  $\underline{d} = (j_1, \dots, j_m)$  we let  $T^{\underline{d}} = (R_T|_{T^{j_1}})^{-1}(T^{\sigma^-(\underline{d})})$ .

We denote  $R_T^{\underline{d}} = R_T|_{T^{\underline{d}}}$  which is always a diffeomorphism onto  $T$ .

Notice that the family of intervals  $T^{\underline{d}}$  is organized by inclusion in the same way as  $\Omega$  is organized by (right side) truncation (the previously introduced tree structure).

If  $T$  is a proper interval, the first return map to  $T$  naturally relates to the first landing to  $T^0$ . Indeed, denoting  $C^d = (R_T^d)^{-1}(T^0)$ , the domain of the first landing map  $L_T$  is easily seen to coincide with the union of the  $C^d$ , and furthermore  $L_T|_{C^d} = R_T^d$ .

Notice that this allows us to relate  $R_T$  and  $R_{T^0}$  since  $R_{T^0} = L_T \circ R_T$ .

**2.1.5. Renormalization.** We say that  $f$  is *renormalizable* if there is an interval  $0 \in T$  and  $m > 1$  such that  $f^m(T) \subset T$  and  $f^j(\text{int } T) \cap \text{int } T = \emptyset$  for  $1 \leq j < m$ . The maximal such interval is called the *renormalization interval of period  $m$* , it has the property that  $f^m(\partial T) \subset \partial T$ .

The set of renormalization periods of  $f$  gives an increasing (possibly empty) sequence of numbers  $m_i$ ,  $i = 1, 2, \dots$ , each related to a unique renormalization interval  $T^{(i)}$  which form a nested sequence of intervals. We include  $m_0 = 1$ ,  $T^{(0)} = I$  in the sequence to simplify the notation.

We say that  $f$  is *finitely renormalizable* if there is a smallest renormalization interval  $T^{(k)}$ . We say that  $f \in \mathcal{F}$  if  $f$  is finitely renormalizable and  $0$  is recurrent but not periodic. We let  $\mathcal{F}_k$  denote the set of maps  $f$  in  $\mathcal{F}$  which are exactly  $k$  times renormalizable.

**2.1.6. Principal nest.** Let  $\Delta_k$  denote the set of all maps  $f$  which have (at least)  $k$  renormalizations and which have an orientation reversing non-attracting periodic point of period  $m_k$  which we denote  $p_k$  (that is,  $p_k$  is the fixed point of  $f^{m_k}|_{T^{(k)}}$  with  $Df^{m_k}(p_k) \leq -1$ ). For  $f \in \Delta_k$ , we denote  $T_0^{(k)} = [-p_k, p_k]$ . We define by induction a (possibly finite) sequence  $T_i^{(k)}$ , such that  $T_{i+1}^{(k)}$  is the component of the domain of  $R_{T_i^{(k)}}$  containing  $0$ . If this sequence is infinite, then either it converges to a point or to an interval.

If  $\cap_i T_i^{(k)}$  is a point, then  $f$  has a recurrent critical point which is not periodic, and it is possible to show that  $f$  is not  $k+1$  times renormalizable. Obviously in this case we have  $f \in \mathcal{F}_k$ , and all maps in  $\mathcal{F}_k$  are obtained in this way: if  $\cap_i T_i^{(k)}$  is an interval, it is possible to show that  $f$  is  $k+1$  times renormalizable.

We can of course write  $\mathcal{F}$  as a disjoint union  $\cup_{i=0}^{\infty} \mathcal{F}_i$ . For a map  $f \in \mathcal{F}_k$  we refer to the sequence  $\{T_i^{(k)}\}_{i=1}^{\infty}$  as the *principal nest*.

It is important to notice that the domain of the first return map to  $T_i^{(k)}$  is always dense in  $T_i^{(k)}$ . Moreover, the next result shows that, outside a very special case, the return map has a hyperbolic structure.

**Lemma 2.1.** *Assume  $T_i^{(k)}$  does not have a non-hyperbolic periodic orbit in its boundary. For all  $T_i^{(k)}$  there exists  $C > 0$ ,  $\lambda > 1$  such that if  $x, f(x), \dots, f^{n-1}(x)$  do not belong to  $T_i^{(k)}$  then  $|Df^n(x)| > C\lambda^n$ .*

This lemma is a simple consequence of a general theorem of Guckenheimer on hyperbolicity of maps of the interval without critical points and non-hyperbolic periodic orbits (Guckenheimer considers unimodal maps with negative Schwarzian derivative, so this applies directly to the case of quadratic maps, the general case is also true by Mañé's Theorem, see [MvS]). Notice that the existence of a non-hyperbolic periodic orbit in the boundary of  $T_i^{(k)}$  depends on a very special combinatorial setting, in particular, all  $T_j^{(k)}$  must coincide (with  $[-p_k, p_k]$ ), and the  $k$ -th renormalization of  $f$  is in fact renormalizable of period 2.

By Lemma 2.1, the maximal invariant of  $f|_{I \setminus T_i^{(k)}}$  is an expanding set, which admits a Markov partition (since  $\partial T_i^{(k)}$  is preperiodic, see also the proof of Lemma 6.1): it is easy to see that it is indeed a Cantor set<sup>3</sup> (except if  $i = 0$  or in the special period 2 renormalization case just described). It follows that the geometry of this Cantor set is well behaved: for instance, its image by any quasimetric map has zero Lebesgue measure.

In particular, one sees that the domain of the first return map to  $T_i^{(k)}$  has infinitely many components (except in the special case above or if  $i = 0$ ) and that its complement has well behaved geometry.

**2.1.7. Lyubich's Regular or Stochastic dichotomy.** A map  $f \in \mathcal{F}_k$  is called *simple* if the principal nest has only finitely many central returns, that is, there are only finitely many  $i$  such that  $R|_{T_i^{(k)}}$  is central. Such maps have many good features, in particular, they are stochastic (this is a consequence of [MN] and [L1]).

In [L3], it was proved that almost every quadratic map is either regular or simple or infinitely renormalizable. It was then shown in [L5] that infinitely renormalizable maps have zero Lebesgue measure, which establishes the Regular or Stochastic dichotomy.

Due to Lyubich's results, we can completely forget about infinitely renormalizable maps, we just have to prove the claimed estimates for almost every simple map.

During our discussion, for notational reasons, we will fix a renormalization level  $\kappa$ , that is, we will only analyze maps in  $\Delta_\kappa$ . This allows us to fix some convenient notation: given  $g \in \Delta_\kappa$  we define  $I_i[g] = T_i^{(\kappa)}[g]$ , so that  $\{I_i[g]\}$  is a sequence of intervals (possibly finite). We use the notation  $R_i[g] = R_{I_i[g]}$ ,  $L_i[g] = L_{I_i[g]}$  and so on (so that the domain of  $R_i[g]$  is  $\cup I_i^j[g]$  and the domain of  $L_i[g]$  is  $\cup C_i^d[g]$ ). When doing phase analysis (working with fixed  $f$ ) we usually drop the dependence on the map and write  $R_i$  for  $R_i[f]$ .

(Notice that, once we fix the renormalization level  $\kappa$ , for  $g \in \Delta_\kappa$ , the notation  $I_i[g]$  stands for  $T_i^{(\kappa)}[g]$ , even if  $g$  is more than  $\kappa$  times renormalizable.)

**2.1.8. Strategy.** To motivate our next steps, let us describe the general strategy behind the proofs of Theorems A and B.

(1) We consider a certain set of non-regular parameters of full measure and describe (in a probabilistic way) the dynamics of the principal nest. This is our phase analysis.

(2) From time to time, we transfer the information from the phase space to the parameter, following the description of the parapuzzle nest which we will make in the next subsection. The rules for this correspondence are referred as Phase-Parameter relation (which is based on the work of Lyubich on complex dynamics of the quadratic family).

(3) This correspondence will allow us to exclude parameters whose critical orbit behaves badly (from the probabilistic point of view) at infinitely many levels of the principal nest. The phase analysis coupled with the Phase-Parameter relation will assure us that the remaining parameters have still full measure.

(4) We restart the phase analysis for the remaining parameters with extra information.

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<sup>3</sup>Dynamically defined Cantor sets with such properties are usually called *regular Cantor sets*.



After many iterations of this procedure we will have enough information to tackle the problems of hyperbolicity and recurrence.

We will first tackle the problem of describing the Phase-Parameter relation, and we will delay all statistical arguments until §3.

A larger outline of this strategy, including the motivation and organization of the statistical analysis, will appear in [AM2].

**2.2. Parameter partition.** Part of our work is to transfer information from the phase space of some map  $f \in \mathcal{F}$  to a neighborhood of  $f$  in the parameter space. This is done in the following way. We consider the first landing map  $L_i$ : the complement of the domain of  $L_i$  is a hyperbolic Cantor set  $K_i = I_i \setminus \cup C_i^d$ . This Cantor set persists in a small parameter neighborhood  $J_i$  of  $f$ , changing in a continuous way. Thus, loosely speaking, the domain of  $L_i$  induces a persistent partition of the interval  $I_i$ .

Along  $J_i$ , the first landing map is topologically the same (in a way that will be clear soon). However the critical value  $R_i[g](0)$  moves relative to the partition (when  $g$  moves in  $J_i$ ). This allows us to partition the parameter piece  $J_i$  in smaller pieces, each corresponding to a region where  $R_i(0)$  belongs to some fixed component of the domain of the first landing map.

**Theorem 2.2** (Topological Phase-Parameter relation). *Let  $f \in \mathcal{F}_\kappa$ . There is a sequence  $\{J_i\}_{i \in \mathbb{N}}$  of nested parameter intervals (the principal parapuzzle nest of  $f$ ) with the following properties.*

- (1)  $J_i$  is the maximal interval containing  $f$  such that for all  $g \in J_i$  the interval  $I_{i+1}[g] = T_{i+1}^{(\kappa)}[g]$  is defined and changes in a continuous way. (Since the first return map to  $R_i[g]$  has a central domain, the landing map  $L_i[g] : \cup C_i^d[g] \rightarrow I_i[g]$  is defined.)
- (2)  $L_i[g]$  is topologically the same along  $J_i$ : there exists homeomorphisms  $H_i[g] : I_i \rightarrow I_i[g]$ , such that  $H_i[g](C_i^d) = C_i^d[g]$ . The maps  $H_i[g]$  may be chosen to change continuously.
- (3) There exists a homeomorphism  $\Xi_i : I_i \rightarrow J_i$  such that  $\Xi_i(C_i^d)$  is the set of  $g$  such that  $R_i[g](0)$  belongs to  $C_{i+1}^d[g]$ .

The homeomorphisms  $H_i$  and  $\Xi_i$  are not uniquely defined, it is easy to see that we can modify them inside each  $C_i^d$  window keeping the above properties. However,  $H_i$  and  $\Xi_i$  are well defined maps if restricted to  $K_i$ .

This fairly standard phase-parameter result can be proved in many different ways. The most elementary proof is probably to use the monotonicity of the quadratic family to deduce the Topological Phase-Parameter relation from Milnor-Thurston's kneading theory by purely combinatorial arguments. Another approach is to use Douady-Hubbard's description of the combinatorics of the Mandelbrot set (restricted to the real line) as does Lyubich in [L3] (see also [AM3] for a more general case).

With this result we can define for any  $f \in \mathcal{F}_\kappa$  intervals  $J_i^j = \Xi_i(I_i^j)$  and  $J_i^d = \Xi_i(I_i^d)$ . From the description we gave it immediately follows that two intervals  $J_{i_1}[f]$  and  $J_{i_2}[g]$  associated to maps  $f$  and  $g$  are either disjoint or nested, and the same happens for intervals  $J_i^j$  or  $J_i^d$ . Notice that if  $g \in \Xi_i(C_i^d) \cap \mathcal{F}_\kappa$  then  $\Xi_i(C_i^d) = J_{i+1}[g]$ .

We will concentrate on the analysis of the regularity of  $\Xi_i$  for the special class of simple maps  $f$ : one of the good properties of the class of simple maps is better

control of the phase-parameter relation. Even for simple maps, however, the regularity of  $\Xi_i$  is not great: there is too much dynamical information contained in it. A solution to this problem is to forget some dynamical information.

**2.2.1. Gape interval.** If  $i > 1$ , we define the *gape interval*  $\tilde{I}_{i+1}$  as follows.

We have that  $R_i|_{I_{i+1}} = L_{i-1} \circ R_{i-1} = R_{i-1}^{\underline{d}} \circ R_{i-1}$  for some  $\underline{d}$ , so that  $I_{i+1} = (R_{i-1}|_{I_i})^{-1}(C_{i-1}^{\underline{d}})$ . We define the gape interval  $\tilde{I}_{i+1} = (R_{i-1}|_{I_i})^{-1}(I_{i-1}^{\underline{d}})$ .

Notice that  $I_{i+1} \subset \tilde{I}_{i+1} \subset I_i$ . Furthermore, for each  $I_i^j$ , the gape interval  $\tilde{I}_{i+1}$  either contains or is disjoint from  $I_i^j$ .

**2.2.2. The Phase-Parameter relation.** As we discussed before, the dynamical information contained in  $\Xi_i$  is entirely given by  $\Xi_i|_{K_i}$ : a map obtained by  $\Xi_i$  by modification inside a  $C_i^{\underline{d}}$  window has still the same properties. Therefore it makes sense to ask about the regularity of  $\Xi_i|_{K_i}$ . As we anticipated before we must erase some information to obtain good results.

Let  $f \in \mathcal{F}_\kappa$  and let  $\tau_i$  be such that  $R_i(0) \in I_i^{\tau_i}$ . We define two Cantor sets,  $K_i^\tau = K_i \cap I_i^{\tau_i}$  which contains refined information restricted to the  $I_i^{\tau_i}$  window and  $\tilde{K}_i = I_i \setminus (\cup I_i^j \cup \tilde{I}_{i+1})$ , which contains global information, at the cost of erasing information inside each  $I_i^j$  window and in  $\tilde{I}_{i+1}$ .

**Theorem 2.3** (Phase-Parameter relation). *Let  $f$  be a simple map. For all  $\gamma > 1$  there exists  $i_0$  such that for all  $i > i_0$  we have*

- PhPa1:**  $\Xi_i|_{K_i^\tau}$  is  $\gamma$ -qs,
- PhPa2:**  $\Xi_i|_{\tilde{K}_i}$  is  $\gamma$ -qs,
- PhPh1:**  $H_i[g]|_{K_i}$  is  $\gamma$ -qs if  $g \in J_i^{\tau_i}$ ,
- PhPh2:** the map  $H_i[g]|_{\tilde{K}_i}$  is  $\gamma$ -qs if  $g \in J_i$ .

The Phase-Parameter relation follows from the work of Lyubich [L3], where a general method based on the theory of holomorphic motions was introduced to deal with this kind of problem. A sketch of the derivation of the specific statement of the Phase-Parameter relation from the general method of Lyubich is given in the Appendix A. The reader can find full details (in a more general context than quadratic maps) in [AM3].

*Remark 2.1.* One of the main reasons why the present work is restricted to the quadratic family is related to the Topological Phase-Parameter relation and the Phase-Parameter relation. The work of Lyubich uses specifics of the quadratic family, specially the fact that it is a full family of quadratic-like maps, and several arguments involved have indeed a global nature (using for instance the combinatorial theory of the Mandelbrot set). Thus we are only able to conclude the Phase-Parameter relation in this restricted setting.

However, the statistical analysis involved in the proofs of Theorem A and B in this work is valid in much more generality. Our arguments suffice (without any changes) for any one-parameter analytic family of unimodal maps  $f_\lambda$  with the following properties:

- (1) For every  $\lambda$ ,  $f_\lambda$  has a quadratic critical point and negative Schwarzian derivative<sup>4</sup>,

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<sup>4</sup>More generally it is enough to ask that the first return map to a sufficiently small nice interval has negative Schwarzian derivative.

(2) For almost every non-regular parameter  $\lambda$ ,  $f_\lambda$  has all periodic orbits repelling (so that Lemma 2.1 holds), is conjugate to a quadratic simple map, and the Topological Phase-Parameter relation<sup>5</sup> and the Phase-Parameter relation<sup>6</sup> are valid at  $\lambda$ .

The assumption of a quadratic critical point is probably the hardest to remove at this point, so our analysis does not apply, say, for the families  $a - x^{2n}$ ,  $n > 1$ . It is worth to point out that most of the arguments developed in this paper go through for higher criticality. The key missing links are in the starting points of this paper: zero Lebesgue measure of infinitely renormalizable parameters and of finitely renormalizable parameters without exponential decay of geometry (in the sense of [L1]), and growth of moduli of parapuzzle annuli (in the sense of [L3]) for almost every parameter.

### 3. MEASURE AND CAPACITIES

**3.1. Quasisymmetric maps.** If  $X \subset \mathbb{R}$  is measurable, let us denote  $|X|$  its Lebesgue measure. Let us explicit the metric properties of  $\gamma$ -qs maps we will use.

To each  $\gamma$ , there exists a constant  $k \geq 1$  such that for all  $f \in QS(\gamma)$ , for all  $J \subset I$  intervals,

$$\frac{1}{k} \left( \frac{|J|}{|I|} \right)^k \leq \frac{|f(J)|}{|f(I)|} \leq \left( \frac{k|J|}{|I|} \right)^{1/k}.$$

Furthermore  $\lim_{\gamma \rightarrow 1} k(\gamma) = 1$ . So for each  $\epsilon > 0$  there exists  $\gamma > 1$  such that  $k(2\gamma - 1) < 1 + \epsilon/5$ . From now on, once a given  $\gamma$  close to 1 is chosen,  $\epsilon$  will always denote a small number with this property.

**3.2. Capacities and trees.** The  $\gamma$ -capacity of a set  $X$  in an interval  $I$  is defined as follows:

$$p_\gamma(X|I) = \sup_{h \in QS(\gamma)} \frac{|h(X \cap I)|}{|h(I)|}.$$

This geometric quantity is well adapted to our context, since it is well behaved under tree decompositions of sets. In other words, if  $I^j$  are disjoint subintervals of  $I$  and  $X \subset \cup I^j$  then

$$p_\gamma(X|I) \leq p_\gamma(\cup_j I^j|I) \sup_j p_\gamma(X|I^j).$$

**3.3. A measure-theoretical lemma.** Our procedure consists in obtaining successively smaller (but still full-measure) classes of maps for which we can give a progressively refined statistical description of the dynamics. This is done inductively as follows: we pick a class  $X$  of maps (which we have previously shown to have full measure among non-regular maps) and for each map in  $X$  we proceed to describe the dynamics (focusing on the statistical behavior of return and landing maps for deep levels of the principal nest), then we use this information to show that a subset  $Y$  of  $X$  (corresponding to parameters for which the statistical behavior of the *critical orbit* is not anomalous) still has full measure. An example of this parameter exclusion process is done by Lyubich in [L3] where he shows using a probabilistic argument that the class of simple maps has full measure in  $\mathcal{F}$ .

<sup>5</sup>Actually one only needs the Topological Phase-Parameter relation to be valid for all deep enough levels of the principal nest.

<sup>6</sup>In [AM1] it is shown how to work around this condition for most families satisfying condition (1). The results obtained are weaker though, and the statistical analysis is slightly harder.

Let us now describe our usual argument (based on the argument of Lyubich which in turn is a variation of the Borel-Cantelli Lemma). Assume at some point we know how to prove that almost every simple map belongs to a certain set  $X$ . Let  $Q_n$  be a (bad) property that a map may have (usually some anomalous statistical parameter related to the  $n$ -th stage of the principle nest). Suppose we prove that if  $f \in X$  then the probability that a map in  $J_n(f)$  has the property  $Q_n$  is bounded by  $q_n(f)$  which is shown to be summable for all  $f \in X$ . We then conclude that almost every map does not have property  $Q_n$  for  $n$  big enough.

Sometimes we also apply the same argument, proving instead that  $q_n(f)$  is summable where  $q_n(f)$  is the probability that a map in  $J_n^{\tau_n}(f)$  has property  $Q_n$ , (recall that  $\tau_n$  is such that  $f \in J_n^{\tau_n}(f)$ ).

In other words, we apply the following general result.

**Lemma 3.1.** *Let  $X \subset \mathbb{R}$  be a measurable set such that for each  $x \in X$  is defined a sequence  $D_n(x)$  of nested intervals converging to  $x$  such that for all  $x_1, x_2 \in X$  and any  $n$ ,  $D_n(x_1)$  is either equal or disjoint to  $D_n(x_2)$ . Let  $Q_n$  be measurable subsets of  $\mathbb{R}$  and  $q_n(x) = |Q_n \cap D_n(x)|/|D_n(x)|$ . Let  $Y$  be the set of all  $x \in X$  which belong to at most finitely many  $Q_n$ . If  $\sum q_n(x)$  is finite for almost any  $x \in X$  then  $|Y| = |X|$ .*

*Proof.* Let  $Y_n = \{x \in X \mid \sum_{k=n}^{\infty} q_k(x) < 1/2\}$ . It is clear that  $Y_n \subset Y_{n+1}$  and  $|\cup Y_n| = |X|$ .

Let  $Z_n = \{x \in Y_n \mid |Y_n \cap D_m(x)|/|D_m(x)| > 1/2, m \geq n\}$ . It is clear that  $Z_n \subset Z_{n+1}$  and  $|\cup Z_n| = |X|$ .

For  $m \geq n$ , let  $T_n^m = \cup_{x \in Z_n} D_m(x)$ . Let  $K_n^m = T_n^m \cap Q_m$ . Of course

$$|K_n^m| = \int_{T_n^m} q_m \leq 2 \int_{Y_n} q_m.$$

And of course

$$\sum_{m \geq n} \int_{Y_n} q_m \leq \frac{1}{2} |Y_n|.$$

This shows that  $\sum_{m \geq n} |K_n^m| \leq |Y_n|$ , so almost every point in  $Z_n$  belongs to at most finitely many  $K_n^m$ . We conclude then that almost every point in  $X$  belongs to at most finitely many  $Q_m$ .  $\square$

The following obvious reformulation will be often convenient

**Lemma 3.2.** *In the same context as above, assume that we are given sequences  $Q_{n,m}$ ,  $m \geq n$  of measurable sets and let  $Y_n$  be the set of  $x$  belonging to at most finitely many  $Q_{n,m}$ . Let  $q_{n,m}(x) = |Q_{n,m} \cap D_m(x)|/|D_m(x)|$ . Let  $n_0(x) \in \mathbb{N} \cup \{\infty\}$  be such that  $\sum_{m=n}^{\infty} q_{n,m}(x) < \infty$  for  $n \geq n_0(x)$ . Then for almost every  $x \in X$ ,  $x \in Y_n$  for  $n \geq n_0(x)$ .*

In practice, we will estimate the capacity of sets in the phase space: that is, given a map  $f$  we will obtain subsets  $\tilde{Q}_n[f]$  in the phase space, corresponding to bad branches of return or landing maps. We will then show that for some  $\gamma > 1$  we have  $\sum p_\gamma(\tilde{Q}_n[f] | I_n[f]) < \infty$  or  $\sum p_\gamma(\tilde{Q}_n[f] | I_n^{\tau_n}[f]) < \infty$ . We will then use PhPa2 or PhPa1, and the measure-theoretical lemma above to conclude that with total probability among non-regular maps, for all  $n$  sufficiently big,  $R_n(0)$  does not belong to a bad set.

From now on when we prove that almost every non-regular map has some property, we will just say that with total probability (without specifying) such property holds.

(To be strictly formal, we have fixed the renormalization level  $\kappa$  (in particular to define the sequence  $J_n$  without ambiguity), so applications of the measure theoretical argument will actually be used to conclude that for almost every parameter in  $\mathcal{F}_\kappa$  a given property holds. Since almost every non-regular map belongs to some  $\mathcal{F}_\kappa$ , this is equivalent to the statement regarding almost every non-regular parameter.)

#### 4. STATISTICS OF THE PRINCIPAL NEST

**4.1. Decay of geometry.** Let as before  $\tau_n \in \mathbb{Z}$  be such that  $R_n(0) \in I_n^{\tau_n}$ .

An important parameter in our construction will be the scaling factor

$$c_n = \frac{|I_{n+1}|}{|I_n|}.$$

This variable of course changes inside each  $J_n^{\tau_n}$  window, however, not by much. From PhPh1, for instance, we get that with total probability

$$\lim_{n \rightarrow \infty} \sup_{g_1, g_2 \in J_n^{\tau_n}} \frac{\ln(c_n[g_1])}{\ln(c_n[g_2])} = 1.$$

This variable is by far the most important on our analysis of the statistics of return maps. We will often consider other variables (say, return times): we will show that the distribution of those variables is concentrated near some average value. Our estimates will usually give a range of values near the average, and  $c_n$  will play an important role. Due (among other issues) to the variability of  $c_n$  inside the parameter windows, the ranges we select will depend on  $c_n$  up to an exponent (say, between  $1 - \epsilon$  and  $1 + \epsilon$ ), where  $\epsilon$  is a small, but fixed, number. From the estimate we just obtained, for big  $n$  the variability (margin of error) of  $c_n$  will fall comfortably in such range, and we won't elaborate more.

A general estimate on the rates decay of  $c_n$  was obtained by Lyubich: he shows that (for a finitely renormalizable unimodal map with a recurrent critical point),  $c_{n_k}$  decays exponentially (on  $k$ ), where  $n_k - 1$  is the subsequence of non-central levels of  $f$ . For simple maps, the same is true with  $n_k = k$ , as there are only finitely many central returns. Thus we can state:

**Theorem 4.1** (see [L1]). *If  $f$  is a simple map then there exists  $C > 0$ ,  $\lambda < 1$  such that  $c_n < C\lambda^n$ .*

Let us use the following notation for the combinatorics of a point  $x \in I_n$ . If  $x \in I_n^j$  we let  $j^{(n)}(x) = j$  and if  $x \in C_n^{\underline{d}}$  we let  $\underline{d}^{(n)}(x) = \underline{d}$ .

**Lemma 4.2.** *With total probability, for all  $n$  sufficiently big we have*

$$(4.1) \quad p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k |I_n|) < k c_n^{1-\epsilon/2},$$

$$(4.2) \quad p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k |I_n|) < e^{-k c_n^{1+\epsilon/2}}.$$

We also have

$$(4.3) \quad p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k |I_n^{\tau_n}|) < k c_n^{1-\epsilon/2},$$

$$(4.4) \quad p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k |I_n^{\tau_n}|) < e^{-k c_n^{1+\epsilon/2}}.$$

*Proof.* Let us compute the first two estimates.

Since  $I_n^0$  is in the middle of  $I_n$ , we have as a simple consequence of the Real Schwarz Lemma (see [L1] and (4.8) in Lemma 4.5 below) that

$$\frac{c_n}{4} < \frac{|C_n^d|}{|I_n^d|} < 4c_n.$$

As a consequence

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| = m|I_n|) < (4c_n)^{1-\epsilon/3}$$

and we get the estimate (4.1) summing up on  $0 \leq m \leq k$ .

For the same reason, we get that

$$\begin{aligned} p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq m+1|I_n|) \\ < \left(1 - \left(\frac{c_n}{4}\right)^{1+\epsilon/3}\right) p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq m|I_n|). \end{aligned}$$

This implies

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq m|I_n|) \leq \left(1 - \left(\frac{c_n}{4}\right)^{1+\epsilon/3}\right)^m.$$

Estimate (4.2) follows from

$$\begin{aligned} \left(1 - \left(\frac{c_n}{4}\right)^{1+\epsilon/3}\right)^k &< (1 - c_n^{1+\epsilon/2})^k \\ &< ((1 - c_n^{1+\epsilon/2})^{c_n^{-1-\epsilon/2}})^{kc_n^{1+\epsilon/2}} \\ &< e^{-kc_n^{1+\epsilon/2}}. \end{aligned}$$

The two remaining estimates are analogous.  $\square$

Let us now transfer this result (more precisely the second pair of estimates) to the parameter in each  $J_n^{\tau_n}$  window using PhPa1. To do this notice that the measure of the complement of the set of parameters in  $J_n^{\tau_n}$  such that  $c_n^{-1+2\epsilon} < s_n < c_n^{-1-2\epsilon}$  can be estimated by  $2c_n^\epsilon$  for  $n$  big which is summable for all  $\epsilon$  by Theorem 4.1. So we have:

**Lemma 4.3.** *With total probability,*

$$\lim_{n \rightarrow \infty} \frac{\ln(s_n)}{\ln(c_n^{-1})} = 1.$$

The parameter  $s_n$  influences the size of  $c_{n+1}$  in a determinant way.

**Corollary 4.4.** *With total probability,*

$$(4.5) \quad \liminf_{n \rightarrow \infty} \frac{\ln(\ln(c_{n+1}^{-1}))}{\ln(c_n^{-1})} \geq 1.$$

*In particular,  $c_n$  decreases at least torrentially fast.*

*Proof.* It is easy to see (using for instance the Real Schwarz Lemma, see [L1], see also item (4.9) in Lemma 4.5 below) that there exists a constant  $K > 0$  (independent of  $n$ ) such that for each  $\underline{d} \in \Omega$ , both components of  $I_n^{\sigma^+(\underline{d})} \setminus I_n^{\underline{d}}$  have size at least  $(e^K - 1)|I_n^{\underline{d}}|$ . In particular, by induction, if  $R_n(0) \in C_n^{\underline{d}}$  we have that both gaps of  $I_n \setminus C_n^{\underline{d}}$  have size at least  $(e^{Ks_n} - 1)|C_n^{\underline{d}}|$ . Taking the preimage by  $R_n$ , and using the

Real Schwarz Lemma again, we see that  $c_{n+1} < Ce^{Ks_n/2}$  for some constant  $C > 0$  independent of  $n$ . We conclude that

$$\liminf \frac{\ln(c_{n+1}^{-1})}{s_n} \geq \frac{K}{2},$$

and since  $c_n \rightarrow 0$  as  $n \rightarrow \infty$  we have

$$\liminf \frac{\ln(\ln(c_{n+1}^{-1}))}{\ln(s_n)} \geq 1$$

which together with Lemma 4.3 implies (4.5).  $\square$

*Remark 4.1.* In the proof of Corollary 4.4, the constant  $K > 0$  is related to the real bounds. In our situation, since we have decay of geometry, we can actually take  $K \rightarrow \infty$  as  $n \rightarrow \infty$ , so we actually have

$$\frac{\ln(c_{n+1}^{-1})}{s_n} \rightarrow \infty$$

torrentially fast.

**4.2. Fine partitions.** We use Cantor sets  $K_n$  and  $\tilde{K}_n$  to partition the phase space. In many circumstances we are directly concerned with intervals of this partition. However, sometimes we just want to exclude an interval of given size (usually a neighborhood of 0). This size does not usually correspond to (the closure of) a union of gaps, so we instead should consider in applications an interval which is a union of gaps, with approximately the given size <sup>7</sup>. The degree of relative approximation will always be torrentially good (in  $n$ ), so we usually won't elaborate on this. In this section we just give some results which will imply that the partition induced by the Cantor sets are fine enough to allow torrentially good approximations.

The following lemma summarizes the situation. The proof is based on estimates of distortion using the Real Schwarz Lemma and the Koebe Principle (see [L1]) and is very simple, so we just sketch the proof.

**Lemma 4.5.** *The following estimates hold:*

$$(4.6) \quad \frac{|I_n^j|}{|I_n|} = O(\sqrt{c_{n-1}}),$$

$$(4.7) \quad \frac{|I_n^d|}{|I_n^{\sigma^+(d)}|} = O(\sqrt{c_{n-1}}),$$

$$(4.8) \quad \frac{c_n}{4} < \frac{|C_n^d|}{|I_n^d|} < 4c_n,$$

$$(4.9) \quad \frac{|\tilde{I}_{n+1}|}{|I_n|} = O(e^{-s_{n-1}}).$$

---

<sup>7</sup>We need to consider intervals which are union of gaps due to our phrasing of the Phase-Parameter relation, which only gives information about such gaps. However, this is not absolutely necessary, and we could have proceeded in a different way: the proof of the Phase-Parameter relation actually shows that there is a *holonomy map* between phase and parameter *intervals* (and not only Cantor sets) corresponding to a *holomorphic motion* for which we can obtain good qs estimates. While this map is not canonical, the fact that it is a holonomy map for a holomorphic motion with good qs estimates would allow our proofs to work.

*Proof.* (Sketch.) Since  $R_n^{\underline{d}}$  has negative Schwarzian derivative, it immediately follows that the Koebe space<sup>8</sup> of  $C_n^{\underline{d}}$  inside  $I_n^{\underline{d}}$  has at least order  $c_n^{-1}$ .

It is easy to see that  $R_{n-1}|_{I_n}$  can be written as  $\phi \circ f$  where  $\phi$  extends to a diffeomorphism onto  $I_{n-2}$  with negative Schwarzian derivative and thus with very small distortion. Since  $R_{n-1}(I_n^j)$  is contained on some  $C_{n-1}^{\underline{d}}$ , we see that the Koebe space of  $I_n^j$  in  $I_n$  is at least of order  $c_{n-1}^{-1/2}$  which implies (4.6).

Let us now consider an interval  $I_n^{\underline{d}}$ . Let  $I_n^j$  be such that  $R_n^{\sigma^+(\underline{d})}(I_n^{\underline{d}}) = I_n^j$ . We can pullback the Koebe space of  $I_n^j$  inside  $I_n$  by  $R_n^{\sigma^+(\underline{d})}$ , so (4.6) implies (4.7). Moreover, this shows by induction that the Koebe space of  $I_n^{\underline{d}}$  inside  $I_n$  is at least of order  $c_{n-1}^{-|\underline{d}|/2}$ . Since  $R_{n-1}(\tilde{I}_{n+1}) \subset I_{n-1}^{\underline{d}}$  with  $|\underline{d}| = s_{n-1}$ , the Koebe space of  $\tilde{I}_{n+1}$  in  $I_n$  is at least  $c_{n-2}^{-|\underline{d}|/4}$ , which implies (4.9).

It is easy to see that  $R_n^{\underline{d}}|_{I_n^{\underline{d}}}$  can be written as  $\phi \circ f \circ R_n^{\sigma^+(\underline{d})}$ , where  $\phi$  has small distortion. Due to (4.6),  $R_n^{\sigma^+(\underline{d})}|_{I_n^{\underline{d}}}$  also has small distortion, so a direct computation with  $f$  (which is purely quadratic) gives (4.8).  $\square$

In other words, distances in  $I_n$  can be measured with precision  $\sqrt{c_{n-1}}|I_n|$  in the partition induced by  $\tilde{K}_n$ , due to (4.6) and (4.9) (since  $e^{-s_{n-1}} \ll c_{n-1}$ ).

Distances can be measured much more precisely with respect to the partition induced by  $K_n$ , in fact we have good precision in each  $I_n^{\underline{d}}$  scale. In other words, inside  $I_n^{\underline{d}}$ , the central gap  $C_n^{\underline{d}}$  is of size  $O(c_n|I_n^{\underline{d}}|)$  (by (4.8)) and the other gaps have size  $O(\sqrt{c_{n-1}}|C_n^{\underline{d}}|)$  (by (4.7) and (4.8)).

**4.3. Initial estimates on distortion.** To deal with the distortion control we need some preliminary known results. Those estimates are based on the Koebe Principle and the estimates of Lemma 4.5. All needed arguments are already contained in the proof of Lemma 4.5, so we won't get into details.

**Proposition 4.6.** *The following estimates hold:*

- (1) For any  $j$ , if  $R_n|_{I_n^j} = f^k$ ,  $\text{dist}(f^{k-1}|_{f(I_n^j)}) = 1 + O(c_{n-1})$ ,
- (2) For any  $\underline{d}$ ,  $\text{dist}(R_n^{\sigma^+(\underline{d})}|_{I_n^{\underline{d}}}) = 1 + O(\sqrt{c_{n-1}})$ .

We will use the following immediate consequence for the decomposition of certain branches.

**Lemma 4.7.** *With total probability,*

- (1)  $R_n|_{I_n^0} = \phi \circ f$  where  $\phi$  has torrentially small distortion,
- (2)  $R_n^{\underline{d}} = \phi_2 \circ f \circ \phi_1$  where  $\phi_2$  and  $\phi_1$  have torrentially small distortion and  $\phi_1 = R_n^{\sigma^+(\underline{d})}$ .

---

<sup>8</sup>The Koebe space of an interval  $T'$  inside an interval  $T \supset T'$  is the minimum of  $|L|/|T'|$  and  $|R|/|T'|$  where  $L$  and  $R$  are the components of  $T \setminus T'$ . If the Koebe space of  $T'$  inside  $T$  is big, then the Koebe Principle states that a diffeomorphism onto  $T'$  which has an extension with negative Schwarzian derivative onto  $T$  has small distortion. In this case, it follows that the Koebe space of the preimage of  $T'$  inside the preimage of  $T$  is also big.



#### 4.4. Estimating derivatives.

**Lemma 4.8.** *Let  $w_n$  denote the relative distance in  $I_n$  of  $R_n(0)$  to  $\partial I_n \cup \{0\}$ :*

$$w_n = \frac{d(R_n(0), \partial I_n \cup \{0\})}{|I_n|}, \quad \text{where } d(x, X) = \inf_{y \in X} |y - x|.$$

*With total probability,*

$$\limsup_{n \rightarrow \infty} \frac{-\ln(w_n)}{\ln(n)} \leq 1.$$

*In particular  $R_n(0) \notin \tilde{I}_{n+1}$  for all  $n$  large enough.*

*Proof.* This is a simple consequence of PhPa2, using that  $n^{-1-\delta}$  is summable, for all  $\delta > 0$  (using (4.9) to obtain the last conclusion).  $\square$

From now on we suppose that  $f$  satisfies the conclusions of the above lemma.

**Lemma 4.9.** *With total probability,*

$$\limsup_{n \rightarrow \infty} \frac{\sup_{j \neq 0} \ln(\text{dist}(f|_{I_n^j}))}{\ln(n)} \leq 1/2.$$

*Proof.* Denote by  $P_n^d$  a  $|C_n^d|/n^{1+\delta}$  neighborhood of  $C_n^d$ . Notice that the gaps of the Cantor sets  $K_n$  inside  $I_n^d$  which are different from  $C_n^d$  are torrentially (in  $n$ ) smaller than  $C_n^d$ , so we can take  $P_n^d$  as a union of gaps of  $K_n$  up to torrentially small error.

It is clear that if  $h$  is a  $\gamma$ -qs homeomorphism ( $\gamma$  close to 1) then

$$|h(P_n^d \setminus C_n^d)| \leq n^{-1-\delta/2} |h(C_n^d)|$$

Notice that if  $C_n^d$  is contained in  $I_n^j$  with  $j \neq \tau_n$ , then  $P_n^d$  does not intersect  $I_n^{\tau_n}$ . Since the  $C_n^d$  are disjoint,

$$p_\gamma(\cup(P_n^d \setminus C_n^d) | I_n^{\tau_n}) \leq n^{-1-\delta/2}$$

which is summable.

Transferring this estimate to the parameter using PhPa1 we see that with total probability, if  $n$  is sufficiently big, if  $R_n(0)$  does not belong to  $C_n^d$  then  $R_n(0)$  does not belong to  $P_n^d$  as well. In particular, if  $n$  is sufficiently big, the critical point 0 will never be in a  $n^{-1/2-\delta/5} |I_{n+1}^j|$  neighborhood of any  $I_{n+1}^j$  with  $j \neq 0$  (the change from  $n^{-1-\delta}$  to  $n^{-1/2-\delta/5}$  is due to taking the inverse image by  $R_n|_{I_{n+1}}$ , which corresponds, up to torrentially small distortion, to taking a square root, and causes the division of the exponent by two). This implies the required estimate on distortion since  $f$  is quadratic.  $\square$

**Lemma 4.10.** *With total probability,*

$$(4.10) \quad \limsup_{n \rightarrow \infty} \frac{\sup_{d \in \Omega} \ln(\text{dist}(R_n^d))}{\ln(n)} \leq \frac{1}{2}.$$

*In particular, for  $n$  big enough,  $\sup_{d \in \Omega} \text{dist}(R_n^d) \leq 2^n$  and  $|DR_n(x)| > 2$ ,  $x \in \cup_{j \neq 0} I_n^j$ .*

*Proof.* By Lemma 4.7, Lemma 4.9 implies (4.10). If  $j \neq 0$ , by (4.6) of Lemma 4.5 we get that  $|R_n(I_n^j)|/|I_n^j| = |I_n|/|I_n^j| > c_{n-1}^{-1/3}$ , so  $\text{dist}(R_n|_{I_n^j}) \leq 2^n$  implies that for all  $x \in I_n^j$ ,  $|DR_n(x)| > c_{n-1}^{-1/3} 2^{-n} > 2$ .  $\square$

*Remark 4.2.* Lemma 4.9 has also an application for approximation of intervals. This result implies that if  $I_n^j = (a, b)$  and  $j \neq 0$ , we have  $1/2^n < b/a < 2^n$ . As a consequence, for any symmetric (about 0) interval  $I_{n+1} \subset X \subset I_n$ , there exists a symmetric (about 0) interval  $\tilde{X} \subset \tilde{X}$ , which is union of  $I_n^j$  and such that  $|\tilde{X}|/|X| < 2^n$  (approximation by union of  $C_n^d$ , with  $|\tilde{X}|/|X|$  torrentially close to 1, follows more easily from the discussion on fine partitions).

We will also need to estimate derivatives of iterates of  $f$ , and not only of return branches.

**Lemma 4.11.** *With total probability, if  $n$  is sufficiently big and if  $x \in I_n^j$ ,  $j \neq 0$ , and  $R_n|_{I_n^j} = f^K$ , then for  $1 \leq k \leq K$ ,  $|Df^k(x)| > |x|c_{n-1}^3$ .*

*Proof.* First notice that by Lemma 4.8 and Lemma 4.7,  $R_n|_{I_n^0} = \phi \circ f$  with  $|D\phi| > 1$ , provided  $n$  is big enough (since  $\phi$  has small distortion and there is a big macroscopic expansion from  $f(I_n^0)$  to  $R_n(I_n^0)$ ). Also, by Lemma 4.4,  $|I_n|$  decays so fast that  $\prod_{r=1}^n |I_r| > c_{n-1}^{3/2}$  for  $n$  big enough. Finally, by Lemma 4.10, for  $n$  big enough,  $|DR_n(x)| > 1$  for  $x \in I_n^j$ ,  $j \neq 0$ . Let  $n_0$  be so big that if  $n \geq n_0$ , all the above properties hold.

From hyperbolicity of  $f$  restricted to the complement of  $I_{n_0}$  (from Lemma 2.1), there exists a constant  $C > 0$  such that if  $s_0$  is such that  $f^s(x) \notin I_{n_0}^0$  for every  $s_0 \leq s < k$  then  $|Df^{k-s_0}(f^{s_0}(x))| > C$ .

Let us now consider some  $n \geq n_0$ . If  $k = K$ , we have a full return and the result follows from Lemma 4.10.

Assume now  $k < K$ . Let us define  $d(s)$ ,  $0 \leq s \leq k$  such that  $f^s(x) \in I_{d(s)} \setminus I_{d(s)}^0$  (if  $f^s(x) \notin I_0$  we set  $d(s) = -1$ ). Let  $m(s) = \max_{s \leq t \leq k} d(t)$ . Let us define a finite sequence  $\{k_r\}_{r=0}^l$  as follows. We let  $k_0 = 0$  and supposing  $k_r < k$  we let  $k_{r+1} = \max\{k_r < s \leq k | d(s) = m(s)\}$ . Notice that  $d(k_i) < n$  if  $i \geq 1$ , since otherwise  $f^{k_i}(x) \in I_n$  so  $k = k_i = K$  which contradicts our assumption.

The sequence  $0 = k_0 < k_1 < \dots < k_l = k$  satisfies  $n = d(k_0) > d(k_1) > \dots > d(k_l)$ . Let  $\theta$  be maximal with  $d(k_\theta) \geq n_0$ . We have of course

$$|Df^{k-k_\theta}(f^{k_\theta}(x))| > C|Df(f^{k_\theta}(x))|,$$

so if  $\theta = 0$  then  $|Df^k(x)| > |2Cx|$  and we are done.

Assume now  $\theta > 0$ . We have of course

$$|Df^{k-k_\theta}(f^{k_\theta}(x))| > C|Df(f^{k_\theta}(x))| > C|I_{d(k_\theta)+1}|$$

For  $1 \leq r \leq \theta$ , the action of  $f^{k_r-k_{r-1}}$  near  $f^{k_{r-1}}(x)$  is obtained by applying the central component of  $R_{d(k_r)}$  followed by several non-central components of  $R_{d(k_r)}$ . Since  $d(k_r) \geq n_0$ , we can estimate

$$|Df^{k_r-k_{r-1}}(f^{k_{r-1}}(x))| > |DR_{d(k_r)}(f^{k_{r-1}}(x))| > |Df(f^{k_{r-1}}(x))|.$$

For  $r = 1$ , this argument gives  $|Df^{k_1}(x)| \geq |Df(x)|$ , while for  $r > 1$  we can estimate

$$|Df^{k_r-k_{r-1}}(f^{k_{r-1}}(x))| > |Df(f^{k_{r-1}}(x))| > |I_{d(k_{r-1})+1}|.$$

Combining it all we get

$$\begin{aligned}
 |Df^k(x)| &= |Df^{k_1}(x)| \cdot |Df^{k-k_1}(f^{k_1}(x))| \prod_{r=2}^{\theta} |Df^{k_r-k_{r-1}}(f^{k_{r-1}}(x))| \\
 &> |2x| \cdot C \cdot |I_{d(k_\theta)+1}| \prod_{r=2}^{\theta} |I_{d(k_{r-1})+1}| = |2Cx| \prod_{r=1}^{\theta} |I_{d(k_r)+1}| \\
 &\geq |2Cx| \prod_{r=0}^n |I_r| > |x|c_{n-1}^3.
 \end{aligned}$$

□

## 5. SEQUENCES OF QUASISYMMETRIC CONSTANTS AND TREES

**5.1. Preliminary estimates.** From now on, we will need to transfer estimates on the capacity of certain sets from level to level of the principal nest. In order to do so we will need to consider not only  $\gamma$ -capacities with some  $\gamma$  fixed, but different constants for different levels of the principal nest. To do so, we will make use of sequences of constants converging (decreasing) to a given value  $\gamma$ . We recall that  $\gamma$  is some constant very close to 1 such that  $k(2\gamma - 1) < 1 + \epsilon/5$ , with  $\epsilon$  very small.

We define the sequences  $\rho_n = (n+1)/n$  and  $\tilde{\rho}_n = (2n+3)/(2n+1)$ , so that  $\rho_n > \tilde{\rho}_n > \rho_{n+1}$  and  $\lim \rho_n = 1$ . We define the sequence  $\gamma_n = \gamma\rho_n$  and an intermediate sequence  $\tilde{\gamma}_n = \gamma\tilde{\rho}_n$ .

As we know, the generalized renormalization process relating  $R_n$  to  $R_{n+1}$  has two phases, first  $R_n$  to  $L_n$  and then  $L_n$  to  $R_{n+1}$ . The following remarks shows why it is useful to consider the sequence of quasisymmetric constants due to losses related to distortion.

*Remark 5.1.* Let  $S$  be an interval contained in  $I_n^d$ . Using Lemma 4.7 we have  $R_n^d|_S = \psi_2 \circ f \circ \psi_1$ , where the distortion of  $\psi_2$  and  $\psi_1$  are torrentially small and  $\psi_1(S)$  is contained in some  $I_n^j$ ,  $j \neq 0$ . If  $S$  is contained in  $I_n^0$  we may as well write  $R_n|_S = \phi \circ f$ , and the distortion of  $\phi$  is also torrentially small.

In either case, if we decompose  $S$  in  $2km$  intervals  $S_i$  of equal length, where  $k$  is the distortion of either  $R_n^d|_S$  or  $R_n|_S$  and  $m$  is subtorrentially big (say,  $m < 2^n$ ), the distortion obtained restricting to any interval  $S_i$  will be bounded by  $1 + m^{-1}$ . Indeed, in the case  $S \subset I_n^0$ , we have  $\text{dist}(R_n|_{S_i}) \leq \text{dist}(\phi) \text{dist}(f|_{S_i})$ . Now  $k = \text{dist}(R_n|_S) \geq \text{dist}(\phi)^{-1} \text{dist}(f|_S)$ . Since  $f$  is quadratic,

$$\begin{aligned}
 \text{dist}(f|_{S_i}) - 1 &\leq \frac{|S_i|}{|S|} (\text{dist}(f|_S) - 1) \leq \frac{1}{2km} (k \text{dist}(\phi) - 1) \\
 &\leq \frac{\text{dist}(\phi)}{2m}.
 \end{aligned}$$

Since  $\text{dist}(\phi) - 1$  is torrentially small,  $\text{dist}(f|_{S_i}) \leq 1 + (2/3)m^{-1}$  and  $\text{dist}(R_n|_{S_i}) \leq 1 + m^{-1}$ . The case  $S \subset I_n^d$  is entirely analogous, considering  $\text{dist}(R_n^d|_{S_i}) \leq \text{dist}(\psi_2) \text{dist}(f|_{\psi_1(S_i)}) \text{dist}(\psi_1)$ , and using torrentially small distortion of  $\psi_1$  and

$\psi_2$ . The estimate now becomes

$$\begin{aligned} \text{dist}(f|_{\psi_1(S_i)}) - 1 &\leq \frac{|\psi_1(S_i)|}{|\psi_1(S)|} (\text{dist}(f|_{\psi_1(S)}) - 1) \leq \frac{\text{dist}(\psi_1)}{2km} (k \text{dist}(\psi_1) \text{dist}(\psi_2) - 1) \\ &\leq \frac{\text{dist}(\psi_1)^2 \text{dist}(\psi_2)}{2m} \end{aligned}$$

and we conclude again that  $\text{dist}(R_n^d|_{S_i}) \leq 1 + m^{-1}$ .

*Remark 5.2.* Let us fix now  $\gamma$  such that the corresponding  $\epsilon$  is small enough. We have the following estimate for the effect of the pullback of a subset of  $I_n$  by the central branch  $R_n|_{I_n^0}$ . With total probability, for all  $n$  sufficiently big, if  $X \subset I_n$  satisfies

$$p_{\tilde{\gamma}_n}(X|I_n) < \delta \leq n^{-1000}$$

then

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X)|I_{n+1}) < \delta^{1/5}.$$

Indeed, let  $V$  be a  $\delta^{1/4}|I_{n+1}|$  neighborhood of 0. Then  $R_n|_{I_{n+1} \setminus V}$  has distortion bounded by  $2\delta^{1/4}$ .

Let  $W \subset I_n$  be an interval of size  $\lambda|I_n|$ . Of course

$$p_{\tilde{\gamma}_n}(X \cap W|W) < \delta \lambda^{-1-\epsilon}.$$

Let us decompose each side of  $I_{n+1} \setminus V$  as a union of  $n^3 \delta^{-1/4}$  intervals of equal length. Let  $W$  be such an interval. From Lemma 4.8, it is clear that the image of  $W$  covers at least  $\delta^{1/2} n^{-4}|I_n|$ . It is clear then that

$$p_{\tilde{\gamma}_n}(X \cap R_n(W)|R_n(W)) < \delta^{(1-\epsilon)/2} n^{4+4\epsilon}.$$

So we conclude that (since the distortion of  $R_n|_W$  is of order  $1 + n^{-3}$  by Remark 5.1)

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X) \cap W|W) < \delta^{(1-\epsilon)/2} n^5$$

(we use the fact that the composition of a  $\gamma_{n+1}$ -qs map with a map with small distortion in  $\tilde{\gamma}_n$ -qs). Since

$$p_{\gamma_{n+1}}(V|I_{n+1}) < (2\delta^{1/4})^{1-\epsilon},$$

we get the required estimate.

**5.2. More on trees.** Let us see an application of the above remarks.

**Lemma 5.1.** *With total probability, for all  $n$  sufficiently big*

$$p_{\tilde{\gamma}_n}((R_n^d)^{-1}(X)|I_n^d) < 2^n p_{\gamma_n}(X|I_n).$$

*Proof.* Decompose  $I_n^d$  in  $n^{\ln(n)}$  intervals of equal length, say,  $\{W_i\}_{i=1}^{n^{\ln(n)}}$ . Then by Lemma 4.10,  $|R_n^d(W_i)| > n^{-2 \ln n}|I_n|$ , so we get

$$p_{\gamma_n}(R_n^d(W_i) \cap X|R_n^d(W_i)) < n^{4 \ln(n)} p_{\gamma_n}(X|I_n).$$

Applying Remark 5.1, we see that

$$p_{\tilde{\gamma}_n}((R_n^d)^{-1}(X) \cap W_i|W_i) < n^{4 \ln(n)} p_{\gamma_n}(X|I_n),$$

(we use the fact that the composition of a  $\tilde{\gamma}_n$ -qs map with a map with small distortion is  $\gamma_n$ -qs) which implies the desired estimate.  $\square$

By induction we get:

**Lemma 5.2.** *With total probability, for  $n$  is big enough, if  $X_1, \dots, X_m \subset \mathbb{Z} \setminus \{0\}$*

$$\begin{aligned} p_{\tilde{\gamma}_n}(\underline{d}^{(n)})(x) &= (j_1, \dots, j_m, \dots, j_{|\underline{d}^{(n)}(x)|}), j_i \in X_i, 1 \leq i \leq m|I_n) \\ &\leq 2^{mn} \prod_{i=1}^m p_{\gamma_n}(j^{(n)}(x) \in X_i | I_n). \end{aligned}$$

The following is an obvious variation of the previous lemma fixing the start of the sequence.

**Lemma 5.3.** *With total probability, for  $n$  is big enough, if  $X_1, \dots, X_m \subset \mathbb{Z} \setminus \{0\}$ , and if  $\underline{d} = (j_1, \dots, j_k)$  we have*

$$\begin{aligned} p_{\tilde{\gamma}_n}(\underline{d}^{(n)})(x) &= (j_1, \dots, j_k, j_{k+1}, \dots, j_{k+m}, \dots, j_{|\underline{d}^{(n)}(x)|}), j_{i+k} \in X_i, 1 \leq i \leq m|I_n^{\underline{d}}) \\ &\leq 2^{mn} \prod_{i=1}^m p_{\gamma_n}(j^{(n)}(x) \in X_i | I_n). \end{aligned}$$

In particular, with  $\underline{d} = (\tau_n)$ ,

$$\begin{aligned} p_{\tilde{\gamma}_n}(\underline{d}^{(n)})(x) &= (\tau_n, j_1, \dots, j_m, j_{m+1}, \dots, j_{|\underline{d}^{(n)}(x)|}), j_i \in X_i, 1 \leq i \leq m|I_n^{\tau_n}) \\ &\leq 2^{mn} \prod_{i=1}^m p_{\gamma_n}(j^{(n)}(x) \in X_i | I_n). \end{aligned}$$

The last part of the above lemma will be often necessary in order to apply PhPa1. Sometimes we are more interested in the case where the  $X_i$  are all equal.

Let  $Q \subset \mathbb{Z} \setminus \{0\}$ . Let  $Q(m, k)$  denote the set of  $\underline{d} = (j_1, \dots, j_m)$  such that  $\#\{1 \leq i \leq m, j_i \in Q\} \geq k$ .

Define  $q_n(m, k) = p_{\tilde{\gamma}_n}(\cup_{\underline{d} \in Q(m, k)} I_n^{\underline{d}} | I_n)$ .

Let  $q_n = p_{\gamma_n}(\cup_{j \in Q} I_n^j | I_n)$ .

**Lemma 5.4.** *With total probability, for  $n$  large enough,*

$$(5.1) \quad q_n(m, k) \leq \binom{m}{k} (2^n q_n)^k.$$

*Proof.* We have the following recursive estimates for  $q_n(m, k)$ :

- (1)  $q_n(1, 0) = 1$ ,  $q_n(1, 1) \leq q_n \leq 2^n q_n$ , and  $q_n(m+1, 0) \leq 1$  for  $m \geq 1$ ,
- (2)  $q_n(m+1, k+1) \leq q_n(m, k+1) + 2^n q_n q_n(m, k)$ .

Indeed, (1) is completely obvious and if  $(j_1, \dots, j_{m+1}) \in Q(m+1, k+1)$  then either  $(j_1, \dots, j_m) \in Q(m, k+1)$  or  $(j_1, \dots, j_m) \in Q(m, k)$  and  $j_{m+1} \in Q$ , so (2) follows from Lemma 5.1. It is clear that (1) and (2) imply by induction (5.1).  $\square$

We recall that by Stirling Formula,

$$\binom{m}{qm} < \frac{m^{qm}}{(qm)!} < \left(\frac{3}{q}\right)^{qm}.$$

So we can get the following estimate. For  $q \geq q_n$ ,

$$(5.2) \quad q_n(m, (6 \cdot 2^n)qm) < \left(\frac{1}{2}\right)^{(6 \cdot 2^n)qm}.$$

It is also used in the following form. If  $q^{-1} > 6 \cdot 2^n$  (it is usually the case, since  $q$  will be torrentially small)

$$(5.3) \quad \sum_{k > q^{-2}} q_n(k, (6 \cdot 2^n)qk) < 2^{-n} q^{-1} \left(\frac{1}{2}\right)^{(6 \cdot 2^n)q^{-1}}.$$

This can be interpreted as a large deviation estimate in this context.

## 6. ESTIMATES ON TIME

Our aim in this section is to estimate the distribution of return times to  $I_n$ : they are concentrated around  $c_{n-1}^{-1}$  up to an exponent close to 1.

The basic estimate is a large deviation estimate which is proven in the next subsection (Corollary 6.5) and states that for  $k \geq 1$  the set of branches with time larger then  $kc_n^{-4}$  has capacity less then  $e^{-k}$ .

**6.1. A Large Deviation lemma for times.** Let  $r_n(j)$  be such that  $R_n|_{I_n^j} = f^{r_n(j)}$ . We will also use the notation  $r_n(x) = r_n(j^{(n)}(x))$ , the  $n$ -th return time of  $x$  (there should be no confusion for the reader, since we consistently use  $j$  for an integer index and  $x$  for a point in the phase space).

Let

$$A_n(k) = p_{\gamma_n}(r_n(x) \geq k | I_n)$$

Since  $f$  restricted to the complement of  $I_{n+1}$  is hyperbolic, from Lemma 2.1, it is clear that  $A_n(k)$  decays exponentially with  $k$ :

**Lemma 6.1.** *With total probability, for all  $n > 0$ , there exists  $C > 0$ ,  $\lambda > 1$  (which depend on  $n$ ) such that  $A_n(k) < C\lambda^k$ .*

*Proof.* Consider a Markov partition for  $f|_{I \setminus I_{n+1}}$ , that is, a finite union of intervals  $M_1, \dots, M_m$  such that

- (1)  $\cup_{i=1}^m M_i = I \setminus I_{n+1}$ ,
- (2) For every  $1 \leq i \leq m$ ,  $f|_{M_i}$  is a diffeomorphism,
- (3)  $f(\cup_{i=1}^m \partial M_i) \subset \cup_{i=1}^m \partial M_i$ .

It is easy to see that such a Markov partition also satisfies

- (4) For every  $1 \leq i \leq m$ , either

$$f(M_i) = \bigcup_{M_j \subset f(M_i)} M_j \quad \text{or} \quad f(M_i) = I_{n+1} \cup \bigcup_{M_j \subset f(M_i)} M_j.$$

(To construct such Markov partition, notice first that the boundary of  $I_{n+1}$  is preperiodic to a periodic orbit  $q$  (of period  $p$ ). In particular we have  $f^s(\partial I_{n+1}) = q$  for some integer  $s > p$ . Let  $K$  be the (finite) set of all  $x$  which never enter  $\text{int } I_{n+1}$  and such that  $f^j(x) = q$  for some  $j \leq s$ . Since  $I_{n+1}$  is nice,  $\partial I_{n+1} \subset K$ , and since  $s > p$ ,  $q \in K$ . In particular  $K$  is forward invariant. It is easy to see that the connected components of  $I \setminus (K \cup I_{n+1})$  form a Markov partition of  $I \setminus I_{n+1}$ .)

It follows that if  $f^j(x) \in \cup_{i=1}^m \text{int } M_i$ ,  $0 \leq j \leq k$  then there exists a unique interval  $x \in M^k(x)$  such that  $f^k|_{M^k(x)}$  is a diffeomorphism onto some  $M_j$ . Notice that if  $k \geq 1$ ,  $f(M^k(x)) = M^{k-1}(f(x))$ .

By Lemma 2.1, if  $y \in M^k(x)$ ,  $|Df^k(y)|$  is exponentially big in  $k$ . In particular,  $\sum_{j=0}^{k-1} |f^j(M^k(x))| < C'$  for some constant  $C' > 0$  independent of  $M^k(x)$ . Since  $f$  is  $C^2$ ,  $\text{dist}(f|_{M^k(x)})$  is uniformly bounded in  $k$ . Notice that the bounds on distortion

depend on  $n$ . (An alternative to this classical argument is to obtain the bounded distortion from the negative Schwarzian derivative).

By Lemma 2.1 again, the set of points  $x \in I$  which never enter  $I_{n+1}$  has empty interior: for every  $T \subset I$  there is an iterate  $f^r(T)$  which intersects  $I_{n+1}$  (otherwise the exponentially growing intervals  $f^r(T) \subset I$  would eventually become bigger than  $I$ ). So there exists  $r > 0$  such that, for every  $M_j$ , there exists  $x \in M_j$  and  $t_j < r$  with  $f^{t_j}(x) \in \text{int } I_{n+1}$ . It follows that there exists an interval  $E_j \subset M_j$  such that  $f^{t_j}(E_j) \subset \text{int } I_{n+1}$ .

Fixing some  $M^k(x)$  with  $f^k(M^k(x)) = M_j$ , let  $E^k(x) = (f^k|_{M^k(x)})^{-1}(E_j)$ . By bounded distortion, it follows that  $|E^k(x)|/|M^k(x)|$  is uniformly bounded from below independently of  $M^k(x)$ . In particular,  $p_{2\gamma}(M^k(x) \setminus E^k(x)|M^k(x)) < \lambda$  for some constant  $\lambda < 1$ .

Let  $M^k$  be the union of the  $M^k(x)$  and  $E^k$  be the union of the  $E^k(x)$ . Then  $M^{k+r} \cap E^k = \emptyset$ . In particular,  $p_{2\gamma}(M^{(k+1)r}|I) < \lambda p_{2\gamma}(M^{kr}|I)$ .

We conclude that  $p_{2\gamma}(M^k|I_n) < C\lambda^{k/r}$  for some constant  $C > 0$ . If  $k > r_n(0)$ , then  $M^k \cap I_n$  contains the set of points  $x \in I_n$  such that  $f^j(x) \notin I_n$ ,  $1 \leq j \leq k$ , that is, all points  $x \in I_n$  with  $r_n(x) > k$ . Adjusting  $C$  and  $\lambda$  if necessary, we have  $A_n(k) < C\lambda^k$ .  $\square$

*Remark 6.1.* It turns out that  $\lambda$  depends strongly on  $n$ . Indeed, it is possible to show that  $\lambda$  is torrentially close to 1. The argument above does not give any estimate on the behavior of  $\lambda$  as  $n$  grows, but it will be used below as the basis of an inductive argument which will give explicit estimates on  $\lambda$  for  $n$  big.

Let  $\zeta_n$  be the maximum  $\zeta \leq c_{n-1}$  such that for all  $k \geq \zeta^{-1}$  we have

$$(6.1) \quad A_n(k) \leq e^{-\zeta k}$$

and finally let  $\alpha_n = \min_{1 \leq m \leq n} \zeta_m$ .

Our main result in this section is to estimate  $\alpha_n$ . We will show that with total probability, for  $n$  big we have  $\alpha_{n+1} \geq c_n^4$ . For this we will have to do a simultaneous estimate for landing times, which we define now.

Let  $l_n(\underline{d})$  be such that  $L_n|_{I_n^{\underline{d}}} = f^{l_n(\underline{d})}$ . We will also use the notation  $l_n(x) = l_n(\underline{d}^{(n)}(x))$ .

Let us define

$$(6.2) \quad B_n(k) = p_{\tilde{\gamma}_n}(l_n(x) > k|I_n).$$

$$(6.3) \quad B_n^{\tau_n}(k) = p_{\tilde{\gamma}_n}(l_n(x) > k + r_n(\tau_n)|I_n^{\tau_n}).$$

**Lemma 6.2.** *If  $k > c_n^{-3/2} \alpha_n^{-3/2}$  then*

$$(6.4) \quad B_n(k) < e^{-c_n^{3/2} \alpha_n^{3/2} k},$$

and

$$(6.5) \quad B_n^{\tau_n} < e^{-c_n^{-3/2} \alpha_n^{3/2} k}.$$

*Proof.* Let us first show (6.4). Let  $k > c_n^{-3/2} \alpha_n^{-3/2}$  be fixed. Let  $m_0 = \alpha_n^{3/2} k$ .

Notice that by Lemma 4.2

$$(6.6) \quad p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq m_0|I_n) \leq e^{-c_n^{5/4} \alpha_n^{3/2} k}.$$

Fix now  $m < m_0$ . Let us estimate

$$(6.7) \quad p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) > k|I_n).$$

For each  $\underline{d} = (j_1, \dots, j_m)$  we can associate a sequence of  $m$  positive integers  $r_i$  such that  $r_i \leq r_n(j_i)$  and  $\sum r_i = k$ . The average value of  $r_i$  is at least  $k/m$  so we conclude that

$$(6.8) \quad \sum_{r_i \geq k/2m} r_i > k/2.$$

Recall also that

$$(6.9) \quad \frac{k}{2m} > \frac{1}{(2\alpha_n^{3/2})} > \alpha_n^{-1}.$$

Given a sequence of  $m$  positive integers  $r_i$  as above we can do the following estimate using Lemma 5.2

$$(6.10) \quad \begin{aligned} p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, \dots, j_m), r_n(j_i) \geq r_i | I_n) \\ \leq 2^{mn} \prod_{j=1}^m p_{\gamma_n}(r_n(x) \geq r_j | I_n) \\ \leq 2^{mn} \prod_{r_j \geq \alpha_n^{-1}} p_{\gamma_n}(r_n(x) \geq r_j | I_n) \\ \leq 2^{mn} \prod_{r_j \geq k/2m} e^{-\alpha_n r_j} \\ \leq 2^{mn} e^{-\alpha_n k/2}. \end{aligned}$$

The number of sequences of  $m$  positive integers  $r_i$  with sum  $k$  is

$$(6.11) \quad \begin{aligned} \binom{k+m-1}{m-1} &\leq \frac{1}{(m-1)!} (k+m-1)^{m-1} \\ &\leq \frac{1}{m!} (k+m)^m \leq \left(\frac{2ek}{m}\right)^m. \end{aligned}$$

Notice that

$$(6.12) \quad \begin{aligned} 2^{mn} \left(\frac{2ek}{m}\right)^m &\leq \left(\frac{2^{n+3}k}{m}\right)^{\frac{m}{k2^{n+3}}} k2^{n+3} \\ &\leq \left(\frac{2^{n+3}k}{m_0}\right)^{\frac{m_0}{k2^{n+3}}} k2^{n+3} \quad (\text{since } x^{1/x} \text{ is decreases for } x > e) \\ &\leq \left(\frac{2^{n+3}}{\alpha_n^{3/2}}\right)^{m_0} \leq e^{\alpha_n^{5/4}k}. \end{aligned}$$

So we can finally estimate

$$(6.13) \quad \begin{aligned} p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) \geq k | I_n) &\leq 2^{mn} \left(\frac{2ek}{m}\right)^m e^{-\alpha_n k/2} \\ &< e^{(\alpha_n^{1/4} - 1/2)\alpha_n k}. \end{aligned}$$



Summing up on  $m$  we get

$$\begin{aligned}
 (6.14) \quad p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| < m_0, l_n(x) \geq k | I_n) \\
 &\leq m_0 e^{(\alpha_n^{1/4} - 1/2)\alpha_n k} \\
 &< e^{(2\alpha_n^{1/4} - 1/2)\alpha_n k} \quad (\text{since } \frac{\ln(m_0)}{k} \leq \frac{\ln(k)}{k} \leq \alpha_n^{5/4}) \\
 &\leq e^{-\alpha_n k/3}.
 \end{aligned}$$

As a direct consequence we get

$$(6.15) \quad B_n(k) < e^{-\alpha_n k/3} + e^{-c_n^{5/4} \alpha_n^{3/2} k} < e^{-c_n^{3/2} \alpha_n^{3/2} k},$$

concluding the proof of (6.4).

For the proof of (6.5) one proceeds analogously. Take  $k$  and  $m_0$  as before. By Lemma 4.7 one gets

$$(6.16) \quad p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq m_0 | I_n^{\tau_n}) \leq e^{-c_n^{5/4} \alpha_n^{3/2} k}.$$

For any  $m < m_0$ , if  $\underline{d} = (\tau_n, j_1, \dots, j_m)$  and  $l_n(\underline{d}) > k + r_n(\tau_n)$  then there exists  $r_i \leq r_n(j_i)$  with  $\sum_{i=1}^m r_i = k$ . Repeating the argument of (6.10) (and using Lemma 5.3 instead of Lemma 5.2) one gets, for any such sequence  $r_1, \dots, r_m$ ,

$$(6.17) \quad p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (\tau_n, j_1, \dots, j_m), r_n(j_i) \geq r_i | I_n^{\tau_n}) \leq 2^{mn} e^{-\alpha_n k/2}.$$

The previous combinatorial estimate can be applied again to obtain

$$(6.18) \quad p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m+1, l_n(x) > k + r_n(\tau_n) | I_n^{\tau_n}) < e^{(\alpha_n^{1/4} - 1/2)\alpha_n k}.$$

Summing up (6.18) on  $m < m_0$  and using (6.16) we obtain estimate (6.5).  $\square$

Let  $v_n = r_n(0)$  be the return time of the critical point.

**Lemma 6.3.** *With total probability, for  $n$  large enough,*

$$v_{n+1} < c_n^{-2} \alpha_n^{-2} / 2.$$

*Proof.* By the definition of  $\alpha_n$  and PhPa2, it follows that with total probability, for  $n$  large enough,

$$r_n(\tau_n) < c_{n-1}^{-1} \alpha_n^{-1}.$$

Recall that  $\underline{d}^{(n)}(0)$  is such that  $R_n(0) \in C_n^{\underline{d}^{(n)}(0)}$ . Using Lemma 6.2, more precisely estimate (6.5), together with PhPa1, we get with total probability, for  $n$  large enough,

$$l_n(\underline{d}^{(n)}(0)) - r_n(\tau_n) < n \alpha_n^{-3/2} c_n^{-3/2},$$

and thus

$$v_{n+1} < v_n + c_{n-1}^{-1} \alpha_n^{-1} + n \alpha_n^{-3/2} c_n^{-3/2} < v_n + \alpha_n^{-2} c_n^{-2} / 4.$$

Notice that  $\alpha_n$  decreases monotonically, thus for  $n_0$  big enough and for  $n > n_0$ ,

$$v_{n+1} < v_{n_0} + \sum_{k=n_0}^n \alpha_k^{-2} c_k^{-2} / 4 < v_{n_0} + \alpha_n^{-2} c_n^{-2} / 3.$$

which for  $n$  big enough implies  $v_{n+1} < c_n^{-2} \alpha_n^{-2} / 2$ .  $\square$

**Lemma 6.4.** *With total probability, for  $n$  large enough,*

$$\alpha_{n+1} \geq \min\{\alpha_n^4, c_n^4\}.$$

*Proof.* Let  $k \geq \max\{\alpha_n^{-4}, c_n^{-4}\}$ . From Lemma 6.3 one immediately sees that if  $r_{n+1}(j) \geq k$  then  $R_n(I_{n+1}^j)$  is contained on some  $C_n^{\underline{d}}$  with  $l_n(\underline{d}) = r_{n+1}(j) - v_n \geq k/2 \geq n\alpha_n^{-3/2}c_n^{-3/2}$ .

Applying Lemma 6.2 we have  $B_n(k/2) < e^{-\alpha_n^{3/2}c_n^{3/2}k/2}$ .  
Applying Remark 5.2 we get

$$A_{n+1}(k) < e^{-k\alpha_n^{3/2}c_n^{3/2}/200} < e^{-k\min\{\alpha_n^4, c_n^4\}}.$$

□

Since  $c_n$  decreases torrentially, we get

**Corollary 6.5.** *With total probability, for  $n$  large enough  $\alpha_{n+1} \geq c_n^4$ .*

*Remark 6.2.* In particular, using Lemma 6.3, for  $n$  big,  $v_n < c_{n-1}^{-2}\alpha_{n-1}^{-2}/2 < c_{n-1}^{-4}$ .

## 6.2. Consequences.

**Lemma 6.6.** *With total probability, for all  $n$  sufficiently large we have*

$$(6.19) \quad p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-1+\epsilon}|I_n) < c_n^{\epsilon/2},$$

$$(6.20) \quad p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-1-5\epsilon/3}|I_n) \leq e^{-c_n^{-\epsilon/4}},$$

$$(6.21) \quad p_{\tilde{\gamma}_n}(l_n(x) - r_n(x) < c_n^{-1+\epsilon}|I_n^{\tau_n}) < c_n^{\epsilon/2},$$

$$(6.22) \quad p_{\tilde{\gamma}_n}(l_n(x) - r_n(x) > c_n^{-1-5\epsilon/3}|I_n^{\tau_n}) \leq e^{-c_n^{-\epsilon/4}}.$$

*Proof.* We will concentrate on estimates (6.19) and (6.20), since (6.21) and (6.22) are analogous.

We have  $l_n(\underline{d}) \geq |\underline{d}|$ , and from Lemma 4.2

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \leq c_n^{-1+\epsilon}|I_n) \leq c_n^{\epsilon/2},$$

which implies (6.19).

On the other hand, by the same lemma,

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq c_n^{-1-\epsilon}|I_n) \leq e^{-c_n^{-\epsilon/2}}.$$

Defining

$$X_m = \bigcup_{\substack{\underline{d}=(j_1, \dots, j_m), \\ r_n(j_m) > c_n^{-\epsilon/2}c_{n-1}^{-4}}} I_n^{\underline{d}}$$

we have

$$p_{\tilde{\gamma}_n}(X_m|I_n) \leq 2^n e^{-c_n^{-\epsilon/2}} < e^{-c_n^{-\epsilon/3}}.$$

Since

$$c_n^{-1-\epsilon}c_n^{-\epsilon/2}c_{n-1}^{-4} < c_n^{-1-5\epsilon/3},$$

we conclude that if  $x$  satisfies  $l_n(x) > c_n^{-1-5\epsilon/3}$  and  $|\underline{d}_n(x)| < c_n^{-1-\epsilon}$  then  $x$  belongs to some  $X_m$  with  $1 \leq m \leq c_n^{-1-\epsilon}$ . So we get

$$p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-1-5\epsilon/3}|I_n) \leq e^{-c_n^{-\epsilon/2}} + c_n^{-1-\epsilon}e^{-c_n^{-\epsilon/3}} < e^{-c_n^{-\epsilon/4}}$$

which implies (6.20). □

**Corollary 6.7.** *With total probability, for all  $n$  sufficiently large we have*

$$(6.23) \quad p_{\gamma_{n+1}}(r_{n+1}(x) < c_n^{-1+\epsilon} | I_{n+1}) < c_n^{\epsilon/10},$$

$$(6.24) \quad p_{\gamma_{n+1}}(r_{n+1}(x) > c_n^{-1-2\epsilon} | I_{n+1}) \leq e^{-c_n^{-\epsilon/5}} \leq c_n^n.$$

*Proof.* Notice that  $r_{n+1}(j) = v_n + l_n(\underline{d})$ , where  $R_n(I_{n+1}^j) \subset C_n^{\underline{d}}$ . By Remark 6.2, we can estimate  $v_n < c_{n-1}^{-10}$ . The distribution of  $r_{n+1}(j) - v_n$  can be then estimated by the distribution of  $l_n(\underline{d})$  from Lemma 6.6, with a slight loss given by Remark 5.2.  $\square$

Using PhPa2 we get

**Lemma 6.8.** *With total probability, for all  $n$  sufficiently big*

$$(6.25) \quad \lim_{n \rightarrow \infty} \frac{\ln(r_n(\tau_n))}{\ln(c_{n-1}^{-1})} = 1.$$

**Corollary 6.9.** *With total probability, for all  $n$  sufficiently large we have*

$$(6.26) \quad p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-1+\epsilon} | I_n^{\tau_n}) \leq c_n^{\epsilon/10},$$

$$(6.27) \quad p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-1-11\epsilon/6} | I_n^{\tau_n}) \leq e^{-c_n^{-\epsilon/5}}.$$

*Proof.* Just use Lemma 6.8 together with estimates (6.21) and (6.22) of Lemma 6.6.  $\square$

**Corollary 6.10.** *With total probability,*

$$\lim_{n \rightarrow \infty} \frac{\ln(v_{n+1})}{\ln(c_n^{-1})} = 1.$$

*Proof.* Notice that  $v_{n+1} = v_n + l_n(\underline{d})$  where  $R_n(0) \in C_n^{\underline{d}}$ . Using Corollary 6.9 and PhPa1 we get  $c_n^{-1+\epsilon} < l_n(\underline{d}) < c_n^{-1-11\epsilon/6}$ . By Remark 6.2,  $v_n < c_{n-1}^{-10}$ , so  $c_n^{-1+\epsilon} < v_{n+1} < c_n^{-1-2\epsilon}$ . Letting  $\epsilon$  go to 0 we get the result.  $\square$

*Remark 6.3.* Using Lemma 4.8, we see that  $|R_n(I_{n+1})| > 2^{-n}|I_n|$ . Since  $|Df(x)| < 4$ ,  $x \in I$ , it follows that  $|DR_n(x)| < 4^{v_n}$ ,  $x \in I_{n+1}$ . In particular, Corollary 6.10 implies that with total probability, for all  $\epsilon > 0$ , for all  $n$  big enough,

$$2^{-n}c_n^{-1} < \frac{|R_n(I_{n+1})|}{|I_{n+1}|} < 4^{v_n} < 4^{c_{n-1}^{-1-\epsilon}},$$

so  $\ln(c_n^{-1}) < c_{n-1}^{-1-2\epsilon}$ . This implies together with Corollary 4.4 that

$$\lim_{n \rightarrow \infty} \frac{\ln(\ln(c_n^{-1}))}{\ln(c_{n-1}^{-1})} = 1,$$

so  $c_n^{-1}$  grows torrentially (and not faster).

## 7. DEALING WITH HYPERBOLICITY

In this section we show by an inductive process that the great majority of branches are reasonably hyperbolic. In order to do that, in the following subsection, we define some classes of branches with “very good” distribution of times and which are not too close to the critical point. The definition of very good distribution of times has an inductive component: they are composition of many very good branches of the previous level. The fact that most branches are very good is related to the validity of some kind of Law of Large Numbers estimate.

### 7.1. Some kinds of branches and landings.

7.1.1. *Standard landings.* Let us define the set of standard landings of level  $n$ ,  $LS(n) \subset \Omega$  as the set of all  $\underline{d} = (j_1, \dots, j_m)$  satisfying the following.

**LS1:** ( $m$  not too small or large)  $c_n^{-1/2} < m < c_n^{-1-2\epsilon}$ ,

**LS2:** (No very large times)  $r_n(j_i) < c_{n-1}^{-14}$  for all  $i$ .

**LS3:** (Short times are sparse in large enough initial segments) For  $c_{n-1}^{-2} \leq k \leq m$

$$\#\{1 \leq i \leq k, r_n(j_i) < c_{n-1}^{-1+2\epsilon}\} < (6 \cdot 2^n) c_{n-1}^{\epsilon/10} k,$$

**LS4:** (Large times are sparse in large enough initial segments) For  $c_n^{-1/n} \leq k \leq m$

$$\#\{1 \leq i \leq k, r_n(j_i) > c_{n-1}^{-1-2\epsilon}\} < (6 \cdot 2^n) e^{-c_{n-1}^{\epsilon/5}} k.$$

**Lemma 7.1.** *With total probability, for all  $n$  sufficiently big we have*

$$(7.1) \quad p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) | I_n) < c_n^{1/3}/2,$$

$$(7.2) \quad p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) | I_n^{\tau_n}) < c_n^{1/3}/2.$$

*Proof.* Let us start with estimate (7.1) (on  $I_n$ ). Let us estimate the complement of the set of landings which violate each item of the definition.

(LS1) This was already estimated before (see Lemma 4.2), an upper bound is  $c_n^{1/3}/3$  (using  $\epsilon$  small).

(LS2) By Corollary 6.5 the  $\gamma_n$ -capacity of  $\{r_n(x) > c_{n-1}^{-14}\}$  is at most  $e^{-c_{n-1}^{-10}} \ll c_n^3$ . Using Lemma 5.1, we see that the  $\tilde{\gamma}_n$ -capacity of the set of  $\underline{d} = (j_1, \dots, j_m)$  with  $r_n(j_i) > c_n^{-14}$  for some  $i \leq c_n^{-1-2\epsilon}$  (in particular for some  $i \leq m$  if  $m$  is as in LS1) is bounded by  $2^n c_n^{-1-2\epsilon} c_n^3 \ll c_n$ .

(LS3) This is a large deviation estimate, so we follow the ideas of §5.2, particularly estimate (5.2). Put  $q = c_{n-1}^{\epsilon/10}$ . By estimate (6.23) of Corollary 6.7, we can estimate the  $\tilde{\gamma}_n$ -capacity corresponding to the violation of LS3 for some fixed  $c_{n-1}^{-2} \leq k \leq c_n^{-1-2\epsilon}$  by

$$\left(\frac{1}{2}\right)^{(6 \cdot 2^n) q k} \leq \left(\frac{1}{2}\right)^{c_{n-1}^{-3/2}} \ll c_n^3.$$

Summing up over  $k \leq c_n^{-1-2\epsilon}$  (and in particular for  $k \leq m$  as in LS1) we get the upper bound  $c_n$ .

(LS4) We use the same method of the previous item. Put  $q = e^{-c_{n-1}^{-\epsilon/5}}$ . By estimate (6.24) of Corollary 6.7, we can bound the  $\tilde{\gamma}_n$ -capacity corresponding to the violation of LS4 for some fixed  $c_n^{-1/n} \leq k \leq c_n^{-1-2\epsilon}$  by

$$\left(\frac{1}{2}\right)^{(6 \cdot 2^n) q k} \ll c_n^3.$$

Summing up over  $k \leq c_n^{-1-2\epsilon}$  (and in particular for  $k \leq m$  as in LS1) we get the upper bound  $c_n$ .

Adding the losses of the four items, we get the estimate (7.1). To get estimate (7.2) (on  $I_n^{\tau_n}$ ), the only item which changes is LS2: we have to be careful since if  $r_n(\tau_n)$  is very large then automatically LS2 is violated for every  $\underline{d}$  which starts by

$\tau_n$ . But this was taken care in Lemma 6.8, and with this observation the estimates are the same.  $\square$

**7.1.2. Very good returns and excellent landings.** Define the set of very good returns,  $VG(n_0, n) \subset \mathbb{Z} \setminus \{0\}$ ,  $n_0, n \in \mathbb{N}$ ,  $n \geq n_0$  by induction as follows. We let  $VG(n_0, n_0) = \mathbb{Z} \setminus \{0\}$  and supposing  $VG(n_0, n)$  already defined, let us define  $LE(n_0, n) \subset LS(n)$  (excellent landings) as the set of standard landings satisfying the following extra condition:

**LE:** (Not very good moments are sparse in large enough initial segments) For all  $c_{n-1}^{-2} < k \leq m$

$$\#\{1 \leq i \leq k, j_i \notin VG(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{1/20} k,$$

And we define  $VG(n_0, n+1)$  as the set of  $j$  such that  $R_n(I_{n+1}^j) = C_n^d$  with  $d \in LE(n_0, n)$  and the satisfying the extra condition:

**VG:** (distant from 0) The distance of  $I_{n+1}^j$  to 0 is bigger than  $c_n^{1/3} |I_{n+1}|$ .

**Lemma 7.2.** *With total probability, for all  $n_0$  sufficiently big and all  $n \geq n_0$ , if*

$$(7.3) \quad p_{\gamma_n}(j^{(n)}(x) \notin VG(n_0, n) | I_n) < c_{n-1}^{1/20}$$

*then*

$$(7.4) \quad p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) | I_n) < c_n^{1/3},$$

$$(7.5) \quad p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) | I_n^{\tau_n}) < c_n^{1/3}.$$

*Proof.* We first use Lemma 7.1 to estimate the  $\tilde{\gamma}_n$ -capacity of branches not in  $LS(n)$  by  $c_n^{1/3}/2$ .

Let  $q = c_{n-1}^{1/20}$ . Using the hypothesis and estimate (5.2) of §5.2 (see also the estimate of the complement of LS3 in Lemma 7.1) we first estimate the  $\tilde{\gamma}_n$ -capacity of the set of landings which violate LE for a specific value of  $k$  with  $k \geq c_{n-1}^{-2}$  by  $(1/2)^{(6 \cdot 2^n) q k}$  and then summing up on  $k$  we get

$$\sum_{k \geq c_{n-1}^{-2}} \left(\frac{1}{2}\right)^{(6 \cdot 2^n) q k} \ll c_n.$$

This argument works both for (7.4) (in  $I_n$ ) and (7.5) (in  $I_n^{\tau_n}$ ).  $\square$

**Lemma 7.3.** *With total probability, for all  $n_0$  sufficiently big and for all  $n \geq n_0$ ,*

$$(7.6) \quad p_{\gamma_n}(j^{(n)}(x) \notin VG(n_0, n) | I_n) < c_{n-1}^{1/20}.$$

*Proof.* It is clear that with total probability, for  $n_0$  sufficiently big and  $n \geq n_0$ , the set of branches  $I_n^j$  at distance at least  $c_{n-1}^{1/3} |I_n|$  of 0 has  $\gamma_n$ -capacity bounded by  $c_{n-1}^{1/8}$ .

For  $n = n_0$ , (7.6) holds (since all branches are very good except the central one). Using Lemma 7.2, if (7.6) holds for  $n$  then (7.4) also holds for  $n$ . Pulling back estimate (7.4) by  $R_n|_{I_{n+1}}$  (using Remark 5.2), we get (7.6) for  $n+1$ . By induction on  $n$ , (7.6) holds for all  $n \geq n_0$ .  $\square$

Using PhPa2 we get (using the measure-theoretical argument of Lemma 3.2)

**Lemma 7.4.** *With total probability, for all  $n_0$  big enough, for all  $n$  big enough,  $\tau_n \in VG(n_0, n)$ .*

**Lemma 7.5.** *With total probability, for all  $n_0$  big enough and for all  $n \geq n_0$ , if  $j \in VG(n_0, n+1)$  then*

$$\frac{1}{2}mc_{n-1}^{-1+2\epsilon} < r_{n+1}(j) < 2mc_{n-1}^{-1-2\epsilon},$$

where as usual,  $m$  is such that  $R_n(I_{n+1}^j) = C_n^{\underline{d}}$  and  $\underline{d} = (j_1, \dots, j_m)$ .

*Proof.* Notice that  $r_{n+1}(j) = v_n + \sum r_n(j_i)$ . To estimate the total time  $r_{n+1}(j)$  from below we use LS3 and get

$$\frac{1}{2}mc_{n-1}^{-1+2\epsilon} < (1 - 6 \cdot 2^n c_n^{\epsilon/10})mc_{n-1}^{-1+2\epsilon} < r_{n+1}(j).$$

To estimate from above, we notice that  $v_n < c_{n-1}^{-4}$  and by LS2 and LS4

$$\sum_{r_n(j_i) > c_{n-1}^{-1-2\epsilon}} r_n(j_i) < 6 \cdot 2^n c_{n-1}^{-14} e^{-c_{n-1}^{-\epsilon/5}} m < m,$$

so

$$r_{n+1}(j) < mc_{n-1}^{-1-2\epsilon} + m + c_{n-1}^{-4} < 2mc_{n-1}^{-1-2\epsilon}.$$

□

*Remark 7.1.* Using LS1 we get the estimate  $c_n^{-1/2} < r_{n+1}(j) < c_n^{-1-3\epsilon}$  for  $j \in VG(n_0, n+1)$ .

Let  $j \in VG(n_0, n+1)$ . We can write  $R_{n+1}|_{I_{n+1}^j} = f^{r_{n+1}(j)}$ , that is, a big iterate of  $f$ . One may consider which proportion of those iterates belong to very good branches of the previous level. More generally, we can truncate the return  $R_{n+1}$ , that is, we may consider  $k < r_{n+1}(j)$  and ask which proportion of iterates up to  $k$  belong to very good branches.

**Lemma 7.6.** *With total probability, for all  $n_0$  big enough and for all  $n \geq n_0$ , the following holds.*

*Let  $j \in VG(n_0, n+1)$ , and let  $\underline{d} = (j_1, \dots, j_m)$  be such that  $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ . Let  $m_k$  be biggest possible with*

$$v_n + \sum_{j=1}^{m_k} r_n(j_i) \leq k$$

*(the amount of full returns to level  $n$  before time  $k$ ) and let*

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in VG(n_0, n)}} r_n(j_i)$$

*(the total time spent in full returns to level  $n$  which are very good before time  $k$ ).*

*Then  $1 - \beta_k/k < c_{n-1}^{1/100}$  if  $k > c_n^{-2/n}$ .*

*Proof.* Let us estimate first the time  $i_k$  which is not spent on non-critical full returns:

$$i_k = k - \sum_{j=1}^{m_k} r_n(j_i).$$

This corresponds exactly to  $v_n$  plus some incomplete part of the return  $j_{m_{k+1}}$ . This part can be bounded by  $c_{n-1}^{-4} + c_{n-1}^{-14}$  (use Corollary 6.10 to estimate  $v_n$  and LS2 to estimate the incomplete part).

Using LS2 we conclude now that

$$m_k > (k - c_{n-1}^{-4} - c_{n-1}^{-14})c_{n-1}^{14} > c_n^{-1/n}$$

so  $m_k$  is not too small.

Let us now estimate the contribution  $h_k$  from full returns  $j_i$  with time higher than  $c_{n-1}^{-1-2\epsilon}$ . Since  $m_k$  is big, we can use LS4 to conclude that the number of such high time returns must be less than  $c_{n-1}^n m_k$ , so their total time is at most  $c_{n-1}^{n-14} m_k$ .

The non very good full returns on the other hand can be estimated by LE (given the estimate on  $m_k$ ), they are at most  $c_{n-1}^{1/21} m_k$ . So we can estimate the total time  $l_k$  of non very good full returns with time less than  $c_{n-1}^{-1-2\epsilon}$  by

$$c_{n-1}^{1/25} c_{n-1}^{-1-2\epsilon} m_k.$$

Since  $m_k$  is big, we can use LS3 to estimate the proportion of branches with not too small time, so we conclude that at most  $c_{n-1}^{\epsilon/11} m_k$  branches are not very good or have time less than  $c_{n-1}^{-1+2\epsilon}$ , so  $\beta_k$  can be estimated from below as

$$(1 - c_{n-1}^{\epsilon/11})c_{n-1}^{-1+2\epsilon} m_k.$$

It is easy to see then that  $i_k/\beta_k \ll c_{n-1}^{1/100}$ ,  $h_k/\beta_k \ll c_{n-1}^{1/100}$ . If  $\epsilon$  is small enough, we also have

$$l_k/\beta_k < 2c_{n-1}^{1/25-4\epsilon} < c_{n-1}^{1/90}.$$

So  $(i_k + h_k + l_k)/\beta_k$  is less than  $c_{n-1}^{1/100}$ . Since  $i_k + h_k + l_k + \beta_k = k$  we have  $1 - \beta_k/k < (i_k + h_k + l_k)/\beta_k$ .  $\square$

**7.1.3. Cool landings.** Let us define the set of cool landings  $LC(n_0, n) \subset \Omega$ ,  $n_0, n \in \mathbb{N}$ ,  $n \geq n_0$  as the set of all  $\underline{d} = (j_1, \dots, j_m)$  in  $LE(n_0, n)$  satisfying

**LC1:** (Starts very good)  $j_i \in VG(n_0, n)$ ,  $1 \leq i \leq c_{n-1}^{-1/30}$ ,

**LC2:** (Short times are sparse in large enough initial segments) For  $c_{n-1}^{-\epsilon/5} \leq k \leq m$

$$\#\{1 \leq i \leq k, r_n(j_i) < c_{n-1}^{-1+2\epsilon}\} < (6 \cdot 2^n) c_{n-1}^{\epsilon/10} k,$$

**LC3:** (Not very good moments are sparse in large enough initial segments)

For all  $c_{n-1}^{-1/30} \leq k \leq m$

$$\#\{1 \leq i \leq k, j_i \notin VG(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{1/60} k,$$

**LC4:** (Large times are sparse in large enough initial segments) For  $c_{n-1}^{-200} \leq k \leq m$

$$\#\{1 \leq i \leq k, r_n(j_i) > c_{n-1}^{-1-2\epsilon}\} < (6 \cdot 2^n) c_{n-1}^{100} k,$$

**LC5:** (Starts with no large times)  $r_n(j_i) < c_{n-1}^{-1-2\epsilon}$ ,  $1 \leq i \leq c_{n-1}^{-\epsilon/5}/2$ .

Notice that LC4 and LC5 overlap, since  $c_{n-1}^{-200} < c_{n-1}^{-\epsilon/5}/2$  as do LC1 and LC3. From this we can conclude that we can control the proportion of large times or non very good times in all moments (and not only for large enough initial segments).

**Lemma 7.7.** *With total probability, for all  $n_0$  sufficiently big and all  $n \geq n_0$ ,*

$$(7.7) \quad p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0, n) | I_n) < c_{n-1}^{1/100}$$

and for all  $n$  big enough

$$(7.8) \quad p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0, n) | I_n^{\tau_n}) < c_{n-1}^{1/100}.$$

*Proof.* We follow the ideas of the proof of Lemma 7.1. Let us start with estimate (7.7). Notice that by Lemmas 7.3 and 7.2 we can estimate the  $\tilde{\gamma}_n$ -capacity of the complement of excellent landings by  $c_n^{1/3}$ . The computations below indicate what is lost going from excellent to cool due to each item of the definition:

(LC1) This is a direct estimate analogous to LS2. By Lemma 7.3, the  $\gamma_n$ -capacity of the complement of very good branches is bounded by  $c_{n-1}^{1/20}$ , so an upper bound for the  $\tilde{\gamma}_n$ -capacity of the set of landings which do not start with  $c_{n-1}^{-1/30}$  very good branches is given by

$$2^n c_{n-1}^{1/20} c_{n-1}^{-1/30} \ll c_{n-1}^{1/100}.$$

(LC2) This is essentially the same large deviation estimate of LS3. We put  $q = c_{n-1}^{\epsilon/10}$ . By estimate (6.23) of Corollary 6.7, the  $\tilde{\gamma}_n$ -capacity of the set of landings violating LC2 for a specific value of  $k$  is bounded by  $(1/2)^{(6 \cdot 2^n)qk}$ , and summing up on  $k$  (see also estimate (5.3)) we get the upper bound

$$\sum_{k \geq c_{n-1}^{-\epsilon/5}} \left(\frac{1}{2}\right)^{(6 \cdot 2^n) c_{n-1}^{\epsilon/10} k} \leq (2^{-n} c_{n-1}^{-\epsilon/10}) \left(\frac{1}{2}\right)^{(6 \cdot 2^n) c_{n-1}^{-\epsilon/10}} \ll c_{n-1}^{1/100}.$$

(LC3) This is analogous to the previous item, we set  $q = c_{n-1}^{1/60}$  and using Lemma 7.3 we get an upper bound

$$\sum_{k \geq c_{n-1}^{-1/30}} \left(\frac{1}{2}\right)^{(6 \cdot 2^n) c_{n-1}^{1/60} k} \leq (2^{-n} c_{n-1}^{-1/60}) \left(\frac{1}{2}\right)^{(6 \cdot 2^n) c_{n-1}^{-1/60}} \ll c_{n-1}^{1/100}.$$

(LC4) As before, we set  $q = c_{n-1}^{100}$  and using estimate (6.24) of Corollary 6.7 we get

$$\sum_{k \geq c_{n-1}^{-200}} \left(\frac{1}{2}\right)^{(6 \cdot 2^n) c_{n-1}^{100} k} \leq (2^{-n} c_{n-1}^{-100}) \left(\frac{1}{2}\right)^{(6 \cdot 2^n) c_{n-1}^{-100}} \ll c_{n-1}^{1/100}.$$

(LC5) This is a direct estimate as LC1, using estimate (6.24) of Corollary 6.7 we get

$$2^n e^{-c_{n-1}^{\epsilon/5}} e^{c_{n-1}^{-\epsilon/5}/2} \ll c_{n-1}^{1/100}.$$

Putting those together, we obtain (7.7). For (7.8), we must be careful to have  $\tau_n \in VG(n_0, n)$  and  $r_n(\tau_n) < c_{n-1}^{-1-2\epsilon}$ , otherwise we would have immediate problems due to LC1 and LC5. But we took care of those properties in Lemmas 7.4 and 6.8, and with this observation the estimates are the same as before.  $\square$

Transferring the result to the parameter, using PhPa1, we get (using the measure-theoretical argument of Lemma 3.2)

**Lemma 7.8.** *With total probability, for all  $n_0$  big enough, for all  $n$  big enough we have  $R_n(0) \in C_n^{\underline{d}}$  with  $\underline{d} \in LC(n_0, n)$ .*

## 7.2. Hyperbolicity.



7.2.1. *Preliminaries.* For  $j \neq 0$ , we define

$$\lambda_n(j) = \inf_{x \in I_n^j} \frac{\ln |DR_n(x)|}{r_n(j)}.$$

And  $\lambda_n = \inf_{j \neq 0} \lambda_n(j)$ . As a consequence of the exponential estimate on distortion for returns (which competes with torrential expansion from the decay of geometry), together with hyperbolicity of  $f$  in the complement of  $I_n^0$  we immediately have the following

**Lemma 7.9.** *With total probability, for all  $n$  sufficiently big,  $\lambda_n > 0$ .*

*Proof.* By Lemma 2.1, there exists a constant  $\tilde{\lambda}_n > 0$  such that each periodic orbit  $p$  of  $f$  whose orbit is entirely contained in the complement of  $I_{n+1}$  must satisfy  $\ln |Df^m(p)| > \tilde{\lambda}_n m$ , where  $m$  is the period of  $p$ . On the other hand, each non-central branch  $R_n|_{I_n^j}$  has a fixed point. By Lemma 4.10,  $\sup \text{dist}(R_n|_{I_n^j}) \leq 2^n$  and of course  $\lim_{j \rightarrow \pm\infty} r_n(j) = \infty$ , so we have

$$\liminf_{j \rightarrow \pm\infty} \lambda_n(j) \geq \tilde{\lambda}_n.$$

On the other hand, for any  $j \neq 0$ ,  $\lambda_n(j) > 0$  by Lemma 4.10, so  $\lambda_n > 0$ .  $\square$

7.2.2. *Good branches.* The “minimum hyperbolicity”  $\liminf \lambda_n$  of the parameters we will obtain will in fact be positive, as it follows from one of the properties of Collet-Eckmann parameters (uniform hyperbolicity on periodic orbits, see [NS]), together with our estimates on distortion.

However our strategy is not to show that the minimum hyperbolicity is positive, but that the typical value of  $\lambda_n(j)$  stays bounded away from 0 as  $n$  grows (and is in fact bigger than  $\lambda_{n_0}/2$  for  $n > n_0$  big). Since we also have to estimate the hyperbolicity of truncated branches it will be convenient to introduce a new class of branches with good hyperbolic properties.

We define the set of good returns  $G(n_0, n) \subset \mathbb{Z} \setminus \{0\}$ ,  $n_0, n \in \mathbb{N}$ ,  $n \geq n_0$  as the set of all  $j$  such that

**G1:** (hyperbolic return)

$$\lambda_n(j) \geq \lambda_{n_0} \frac{1 + 2^{n_0-n}}{2},$$

**G2:** (hyperbolicity in truncated return) for  $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$  we have

$$\inf_{x \in I_n^j} \frac{\ln |Df^k(x)|}{k} \geq \lambda_{n_0} \frac{1 + 2^{n_0-n+1/2}}{2} - c_{n-1}^{2/(n-1)}.$$

Notice that since  $c_n$  decreases torrentially, for  $n$  sufficiently big G2 implies that if  $j$  is good then for  $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$  we have

$$\inf_{x \in I_n^j} \frac{\ln |Df^k(x)|}{k} \geq \lambda_{n_0} \frac{1 + 2^{n_0-n}}{2}.$$

**Lemma 7.10.** *With total probability, for all  $n_0$  big enough and for all  $n > n_0$ ,  $VG(n_0, n) \subset G(n_0, n)$ .*

*Proof.* Let us prove that if G1 is satisfied for all  $j \in VG(n_0, n)$ , then  $VG(n_0, n+1) \subset G(n_0, n+1)$  (notice that by definition of  $\lambda_{n_0}$  the hypothesis is satisfied for  $n_0$ ). Fix  $j \in VG(n_0, n+1)$  and define

$$a_k = \inf_{x \in I_{n+1}^j} \frac{\ln |Df^k(x)|}{k},$$

and let us consider values of  $k$  in the range  $c_n^{-3/n} \leq k \leq r_{n+1}(j)$  (notice that if  $k = r_{n+1}(j)$  belongs to this range by Remark 7.1).

We let (as usual)  $R_n(I_{n+1}^j) \subset C_n^{\underline{d}}$ ,  $\underline{d} = (j_1, \dots, j_m)$ . Notice that by Corollary 6.10,  $v_n < c_{n-1}^{-4} < k$ . Let us say that  $j_i$  was completed before  $k$  if  $v_n + r_n(j_1) + \dots + r_n(j_i) \leq k$ . We let the queue be defined as

$$q_k = \inf_{x \in C_n^{\underline{d}}} \ln |Df^{k-r} \circ f^r(x)|$$

where  $r = v_n + r_n(j_1) + \dots + r_n(j_{m_k})$  with  $j_{m_k}$  the last complete return.

Let us show first that  $|DR_n(x)| > 1$  if  $x \in I_{n+1}^j$ . Indeed, by Lemma 4.7,  $DR_n|_{I_{n+1}} = \phi \circ f$ , where  $\phi$  has small distortion, so by Lemma 4.8,

$$|D\phi(x)| > \frac{|R_n(I_{n+1})|}{2|f(I_{n+1})|} > \frac{2^{-n}|I_n|}{|I_{n+1}|^2},$$

while by VG,  $|Df(x)| = |2x| \geq c_n^{1/3}|I_{n+1}|$ , so  $|DR_n(x)| > c_n^{-1/2}$ .

By Lemma 4.10, any complete return before  $k$  produces some expansion, (that is, the absolute value of the derivative of such return is at least 1). On the other hand,  $-q_k$  can be bounded from above by  $-\ln(c_n c_{n-1}^5)$  using Lemma 4.11. We have

$$-\frac{q_k}{k} \leq \frac{-\ln(c_n c_{n-1}^5)}{c_n^{-3/n}} \ll c_n^{2/n}.$$

Now we use Lemma 7.6 and get

$$\begin{aligned} a_k &> \frac{\beta_k \lambda_{n_0} (1 + 2^{n_0-n})}{k} - \frac{-q_k}{k} \\ &\geq \frac{\lambda_{n_0} (1 + 2^{n_0-n-1/2})}{2} - \frac{-q_k}{k} \end{aligned}$$

which gives G2. If  $k = r_{n+1}(j)$  then  $q_k = 0$ , which gives G1.  $\square$

### 7.2.3. Hyperbolicity in cool landings.

**Lemma 7.11.** *With total probability, if  $n_0$  is sufficiently big, for all  $n$  sufficiently big, if  $\underline{d} \in LC(n_0, n)$  then for all  $c_{n-1}^{-4/(n-1)} < k \leq l_n(\underline{d})$ ,*

$$\inf_{x \in C_n^{\underline{d}}} \frac{\ln |Df^k(x)|}{k} \geq \frac{\lambda_{n_0}}{2}.$$

*Proof.* Fix such  $\underline{d} \in LC(n_0, n)$ , and let as usual  $\underline{d} = (j_1, \dots, j_m)$ . Let

$$a_k = \inf_{x \in C_n^{\underline{d}}} \frac{\ln |Df^k(x)|}{k}.$$

Analogously to Lemma 7.6, we define  $m_k$  as the number of full returns before  $k$ , so that  $m_k$  is the biggest integer such that

$$\sum_{i=1}^{m_k} r_n(j_i) \leq k.$$

We define

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in VG(n_0, n)}} r_n(j_i),$$

(counting the time up to  $k$  spent in complete very good returns) and

$$i_k = k - \sum_{i=1}^{m_k} r_n(j_i).$$

(counting the time in the incomplete return at  $k$ ).

Let us now consider two cases: either all iterates are part of very good returns (that is, all  $j_i$ ,  $1 \leq i \leq m_k$  are very good and if  $i_k > 0$  then  $j_{m_k+1}$  is also very good), or some iterates are not part of very good returns.

*Case 1 (All iterates are part of very good returns).* Since full good returns are very hyperbolic by G1 and very good returns are good, we just have to worry with possibly losing hyperbolicity in the incomplete time. To control this, we introduce the queue

$$q_k = \inf_{x \in C_n^d} \ln |Df^{i_k} \circ f^{k-i_k}(x)|.$$

We have  $-q_k \leq -\ln(c_{n-1}^{1/3} c_{n-1}^5)$  by Lemma 4.11 and VG, using that the incomplete time is in the middle of a very good branch. Let us split again in two cases:  $i_k$  big or otherwise.

*Subcase 1a* ( $i_k \geq c_{n-1}^{-4/(n-1)}$ ). If the incomplete time is big, we can use G2 to estimate the hyperbolicity of the incomplete time (which is part of a very good return):  $q_k/i_k > \lambda_{n_0}/2$ . We have

$$a_k > \lambda_{n_0} \frac{(1 + 2^{n_0-n})}{2} \cdot \frac{k - i_k}{k} + \frac{q_k}{i_k} \cdot \frac{i_k}{k} > \frac{\lambda_{n_0}}{2}.$$

*Subcase 1b* ( $i_k < c_{n-1}^{-4/(n-1)}$ ). If the incomplete time is not big, we can not use G2 to estimate  $q_k$ , but in this case  $i_k$  is much less than  $k$ : since  $k > c_{n-1}^{-4/(n-1)}$ , at least one return was completed ( $m_k \geq 1$ ), and since it must be very good we conclude that  $k > c_{n-1}^{-1/2}$  by Remark 7.1, so for  $n$  big

$$a_k > \lambda_{n_0} \frac{(1 + 2^{n_0-n})}{2} \cdot \frac{k - i_k}{k} - \frac{-q_k}{k} > \frac{\lambda_{n_0}}{2}.$$

*Case 2 (Some iterates are not part of a very good return).* By LC1,  $m_k > c_{n-1}^{-1/30}$ . Notice that by LC2, if  $m_k > c_{n-1}^{-\epsilon/5}$  then

$$k - i_k > c_{n-1}^{-1+2\epsilon} m_k / 2.$$

So it follows that  $m_k > c_{n-1}^{-1/30}$  implies that  $k > c_{n-1}^{-35/34}$  (using small  $\epsilon$ ).

For the incomplete time we have  $-q_k \leq -\ln(c_n c_{n-1}^5) < c_{n-1}^{-1-\epsilon}$ , so  $-q_k/k < c_{n-1}^{1/100}$ .

Arguing as in Lemma 7.6, we split  $k - \beta_k - i_k$  (time of full returns which are not very good) in a part relative to returns with high time (more than  $c_{n-1}^{-1-2\epsilon}$ ) which

we denote  $h_k$  and in a part relative to returns with low time (less than  $c_{n-1}^{-1-2\epsilon}$ ) which we denote  $l_k$ . Using LC4 and LC5 to bound the number of returns with high time, and using LS2 to bound their time, we get

$$h_k < c_{n-1}^{-14} (6 \cdot 2^n) c_{n-1}^{100} m_k,$$

and using LC1 and LC3 we have

$$l_k < c_{n-1}^{-1-2\epsilon} (6 \cdot 2^n) c_{n-1}^{1/60} m_k < c_{n-1}^{-79/80} m_k,$$

provided  $\epsilon$  is small enough.

Since  $k > c_{n-1}^{-1+2\epsilon} m_k / 2$  we have

$$\frac{h_k + l_k}{k} < 4c_{n-1}^{1/85},$$

provided  $\epsilon$  is small enough.

Now if  $i_k < c_{n-1}^{-1-2\epsilon}$  then  $i_k/k < c_{n-1}^{1/80}$  (using  $\epsilon$  small), and if  $i_k > c_{n-1}^{-1-2\epsilon}$  then by LC5,  $m_k \geq e^{c_{n-1}^{-\epsilon/5}} > c_{n-1}^{-n}$ , so by LS2,  $i_k/k < i_k/m_k < c_{n-1}^{-14}/c_{n-1}^{-n}$ . So in both cases  $i_k/k < c_{n-1}^{1/80}$ .

From our estimates on  $i_k$  and on  $h_k$  and  $l_k$  we have  $1 - (\beta_k/k) < c_{n-1}^{1/90}$ . Now very good returns are very hyperbolic, and full returns (even not very good ones) always give derivative at least 1 from Lemma 4.10, so we have the estimate

$$a_k > \lambda_{n_0} \frac{(1 + 2^{n-n_0})}{2} \cdot \frac{\beta_k}{k} - \frac{-q_k}{k} > \frac{\lambda_{n_0}}{2}$$

for  $n$  big. □

## 8. MAIN THEOREMS

**8.1. Proof of Theorem A.** We must show that with total probability,  $f$  is Collet-Eckmann. We will use the estimates on hyperbolicity of cool landings to show that if the critical point always falls in a cool landing then there is uniform control of the hyperbolicity along the critical orbit.

Let

$$a_k = \frac{\ln |Df^k(f(0))|}{k}$$

and  $e_n = a_{v_n-1}$ .

It is easy to see that if  $n_0$  is big enough such that the conclusions of both Lemmas 7.8 and 7.11 are valid, we obtain for  $n$  large enough that

$$e_{n+1} \geq e_n \frac{v_n - 1}{v_{n+1} - 1} + \frac{\lambda_{n_0}}{2} \cdot \frac{v_{n+1} - v_n}{v_{n+1} - 1}$$

and so

$$(8.1) \quad \liminf_{n \rightarrow \infty} e_n \geq \frac{\lambda_{n_0}}{2}.$$

Let now  $v_n - 1 < k < v_{n+1} - 1$ . Define  $q_k = \ln |Df^{k-v_n}(f^{v_n}(0))|$ .

Assume first that  $k \leq v_n + c_{n-1}^{-4/(n-1)}$ . From LC1 we know that  $\tau_n$  is very good, so by LS1 we have  $r_n(\tau_n) > c_{n-1}^{-1/2}$ , so  $k$  is in the middle of this branch (that is,  $v_n \leq k \leq v_n + r_n(\tau_n) - 1$ ). Using that  $|R_n(0)| > |I_n|/2^n$  (see Lemma 4.8), we get by Lemma 4.11 that

$$-q_k < -\ln(2^{-n} c_{n-1}^5) < c_{n-2}^{-1-\epsilon}.$$

Since  $v_n > c_{n-1}^{-1+\epsilon}$  (by Lemma 6.10) we have

$$(8.2) \quad a_k \geq e_n \frac{v_n - 1}{k} - \frac{-q_k}{k} > \left(1 - \frac{1}{2^n}\right) e_n - \frac{1}{2^n}.$$

If  $k > v_n + c_{n-1}^{-4/(n-1)}$ , using Lemma 7.11 we get

$$(8.3) \quad a_k \geq e_n \frac{v_n - 1}{k} + \frac{\lambda_{n_0}}{2} \cdot \frac{k - v_n + 1}{k}.$$

It is clear that estimates (8.1), (8.2) and (8.3) imply that  $\liminf_{k \rightarrow \infty} a_k \geq \lambda_{n_0}/2$  and so  $f$  is Collet-Eckmann.

**8.2. Proof of Theorem B.** We must obtain, with total probability, upper and lower (polynomial) bounds for the recurrence of the critical orbit. It will be easier to first study the recurrence with respect to iterates of return branches, and then estimate the total time of those iterates.

**8.2.1. Recurrence in terms of return branches.** The principle of the phase analysis is very simple: for the essentially Markov process generated by iteration of the non-central branches of  $R_n$ , most orbits (in the qs sense) approach 0 at a polynomial rate before falling in  $I_{n+1}$ . From this we conclude, using the phase-parameter relation, that with total probability the same holds for the critical orbit.

**Lemma 8.1.** *With total probability, for  $n$  big enough and for  $1 \leq i \leq c_{n-1}^{-2}$ ,*

$$\frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} < (1 + 4\epsilon) \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

*Proof.* Notice that due to torrential (and monotonic) decay of  $c_n$ , we can estimate  $|I_n| = c_{n-1}^{1+\delta_n}$ , with  $\delta_n$  decaying torrentially fast.

From Lemma 4.8, we have

$$\frac{\ln |R_n(0)|}{\ln c_{n-1}} < \frac{\ln(2^{-n}|I_n|)}{\ln c_{n-1}} < 1 + 4\epsilon$$

and the result follows for  $i = 1$ .

For  $1 \leq j \leq 2\epsilon^{-1}$ , let  $X_j \subset I_n$  be a  $c_{n-1}^{(1+2\epsilon)(1+j\epsilon)}$  neighborhood of 0. For  $n$  big, we can estimate (due to the relation between  $|I_n|$  and  $c_{n-1}$ )

$$\frac{|X_j|}{|I_n|} < \frac{c_{n-1}^{(1+2\epsilon)(1+j\epsilon)}}{c_{n-1}^{1+2\epsilon}} = c_{n-1}^{j\epsilon(1+2\epsilon)}$$

(we of course consider  $X_j$  as a union of  $C_n^d$ , so that its size is near the required size, the precision is high enough for our purposes due to Remark 4.2).

We have to make sure that the critical point does not land in some  $X_j$  for  $c_{n-1}^{(1-j)\epsilon} < i \leq c_{n-1}^{-j\epsilon}$ . This requirement can be translated on  $R_n(0)$  not belonging to a certain set  $Y_j \subset I_n$  such that

$$Y_j = \bigcup_{c_{n-1}^{(1-j)\epsilon} \leq |d| < c_{n-1}^{-j\epsilon}} (R_n^d)^{-1}(X_j).$$

By Lemma 4.8, it is clear that no  $X_j$  intersects  $I_n^{\tau_n}$ , so we easily get

$$p_\gamma(Y_j | I_n^{\tau_n}) \leq c_{n-1}^{-j\epsilon} c_{n-1}^{(1+\epsilon)j\epsilon} \leq c_{n-1}^{\epsilon^2}$$

and

$$p_\gamma\left(\bigcup_{j=1}^{2\epsilon^{-1}} Y_j | I_n^{\tau_n}\right) < 2\epsilon^{-1} c_{n-1}^{\epsilon^2}.$$

Applying PhPa1, the probability that for some  $1 \leq j \leq 2\epsilon^{-1}$  and  $c_{n-1}^{(1-j)\epsilon} < i \leq c_{n-1}^{-j\epsilon}$  we have  $|R_n^i(0)| < c_{n-1}^{(1+2\epsilon)(1+j\epsilon)}$  is at most  $2\epsilon^{-1} c_{n-1}^{\epsilon^2}$ , which is summable. In particular, with total probability, for  $j$  and  $i$  as above, we have for  $n$  big enough

$$\begin{aligned} \frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} &\leq (1+2\epsilon)(1+j\epsilon) \\ &< (1+4\epsilon)(1+(j-1)\epsilon) < (1+4\epsilon) \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right). \end{aligned}$$

□

**Lemma 8.2.** *With total probability, for  $n$  big enough and for  $c_{n-1}^{-1-\epsilon} < i \leq s_n$ ,*

$$\frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} < (1+4\epsilon) \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

*Proof.* The argument is the same as for the previous lemma, but the decomposition has a slightly different geometry. Let

$$x_j = c_{n-1}^{(1+2\epsilon)(1+(1+\epsilon)^{j+1})},$$

so that

$$\frac{x_j}{|I_n|} < \frac{c_{n-1}^{(1+2\epsilon)(1+(1+\epsilon)^{j+1})}}{c_{n-1}^{1+\epsilon}} < c_{n-1}^{(1+2\epsilon)(1+\epsilon)^{j+1}}.$$

Let  $K$  be biggest with  $x_K > c_n^{1-\epsilon}$ . For  $0 \leq j \leq K$ , let  $X_j \subset I_n$  be a  $x_j$  neighborhood of 0 (approximated as union of  $C_n^d$ , notice that  $x_j > c_n^{1-\epsilon} \gg |I_{n+1}|$  for  $0 \leq j \leq K$ , so the approximation is good enough for our purposes due to Remark 4.2). Let  $Y_j \subset I_n$  be such that

$$Y_j = \bigcup_{c_{n-1}^{-(1+\epsilon)^j} \leq |d| < c_{n-1}^{-(1+\epsilon)^{j+1}}} (R_n^d)^{-1}(X_j).$$

By Lemma 4.8, it is clear that no  $X_j$  intersects  $I_n^{\tau_n}$ , so we easily get

$$p_\gamma(Y_j | I_n^{\tau_n}) \leq c_{n-1}^{-(1+\epsilon)^{j+1}} c_{n-1}^{(1+\epsilon)^{j+2}} \leq c_{n-1}^{\epsilon(1+j\epsilon)}$$

and

$$p_\gamma\left(\bigcup_{j=0}^K Y_j | I_n^{\tau_n}\right) < \sum_{j=0}^{\infty} c_{n-1}^{\epsilon(1+j\epsilon)} = \frac{c_{n-1}^\epsilon}{1 - c_{n-1}^{\epsilon^2}} < c_{n-1}^{\epsilon/2}.$$

Applying PhPa1, the probability that for some  $0 \leq j \leq K$  and

$$c_{n-1}^{-(1+\epsilon)^j} < i \leq c_{n-1}^{-(1+\epsilon)^{j+1}}$$

we have

$$|R_n^i(0)| < c_{n-1}^{(1+2\epsilon)(1+(1+\epsilon)^{j+1})}$$

is at most  $c_{n-1}^{\epsilon/2}$ , which is summable. In particular, with total probability, for  $j$  and  $i$  as above, we have

$$\begin{aligned} \frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} &< (1+2\epsilon)(1+(1+\epsilon)^{j+1}) \\ &< (1+4\epsilon)(1+(1+\epsilon)^j) < (1+4\epsilon) \left( 1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})} \right). \end{aligned}$$

This covers the range  $c_{n-1}^{-1} < i \leq c_{n-1}^{-(1+\epsilon)^{K+1}}$ . For  $c_{n-1}^{-(1+\epsilon)^{K+1}} < i \leq s_n$ , notice that  $R_n^i(0) \notin I_{n+1}$ , so

$$\begin{aligned} \frac{\ln |R_n^i(0)|}{\ln c_{n-1}} &< \frac{\ln(|I_{n+1}|/2)}{\ln c_{n-1}} \\ &< \frac{1+4\epsilon}{1+2\epsilon} \cdot \frac{\ln c_n^{1-\epsilon}}{\ln c_{n-1}} \\ &\leq \frac{1+4\epsilon}{1+2\epsilon} \cdot \frac{\ln x_{K+1}}{\ln c_{n-1}} && \text{(by definition of } K\text{)} \\ &\leq (1+4\epsilon)(1+(1+\epsilon)^{K+1}) \\ &\leq (1+4\epsilon) \left( 1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})} \right). \end{aligned}$$

□

Both cases are summarized below:

**Corollary 8.3.** *With total probability, for  $n$  big enough and for  $1 \leq i \leq s_n$ ,*

$$\frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} < (1+4\epsilon) \left( 1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})} \right).$$

**8.2.2. Total time of full returns.** We must now relate the return times in terms of  $R_n$  to the return times in terms of  $f$ .

For  $1 \leq i \leq s_n$ , let  $k_i$  be such that  $R_n^i(0) = f^{k_i}(0)$ .

**Lemma 8.4.** *With total probability, for  $n$  big enough and for  $c_{n-1}^{-\epsilon} < i \leq s_n$ ,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > (1-3\epsilon) \left( 1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})} \right).$$

*Proof.* By Lemma 7.8,  $R_n(0)$  belongs to a cool landing, so using LC2 (which allows to estimate the average of return times over a large initial segment of cool landings) we get

$$\frac{k_i}{i-1} > c_{n-1}^{-1+3\epsilon}.$$

This immediately gives

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > (1-3\epsilon) + \frac{\ln(i-1)}{\ln(c_{n-1}^{-1})} > (1-3\epsilon) \left( 1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})} \right)$$

□

Using that  $v_n > c_{n-1}^{-1+\epsilon}$  (from Corollary 6.10) and that  $k_i \geq v_n$  we get for  $1 \leq i \leq c_{n-1}^{-\epsilon}$

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} \geq \frac{\ln(v_n)}{\ln(c_{n-1}^{-1})} > \frac{\ln(c_{n-1}^{-1+\epsilon})}{\ln(c_{n-1}^{-1})} > (1-3\epsilon)(1+\epsilon) \geq (1-3\epsilon) \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Together with Lemma 8.4, this gives

**Corollary 8.5.** *With total probability, for  $n$  big enough and for  $1 \leq i \leq s_n$ ,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > (1-3\epsilon) \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

8.2.3. *Upper and lower bounds.* Notice that  $|R_n(0)| = |f^{v_n}(0)| \leq c_{n-1}$ , so using Lemma 6.10 we get

$$\limsup_{n \rightarrow \infty} \frac{-\ln |f^n(0)|}{\ln(n)} \geq \limsup_{n \rightarrow \infty} \frac{-\ln |f^{v_n}(0)|}{\ln(v_n)} \geq \limsup_{n \rightarrow \infty} \frac{-\ln(c_{n-1})}{\ln(v_n)} \geq 1.$$

Let now  $v_n \leq k < v_{n+1}$ . If  $|f^k(0)| < k^{-1-10\epsilon}$  then Lemma 6.10 implies that  $f^k(0) \in I_n$  and so  $k = k_i$  for some  $i$ . It follows from Corollaries 8.3 and 8.5 that

$$|f^{k_i}(0)| > k_i^{-1-10\epsilon}.$$

Varying  $\epsilon$  we get

$$\limsup_{n \rightarrow \infty} \frac{-\ln |f^n(0)|}{\ln(n)} \leq 1.$$

## APPENDIX A. SKETCH OF THE PROOF OF THE PHASE-PARAMETER RELATION

The proof of the Phase-Parameter relation uses ideas from complex analysis. We will provide a sketch of the proof assuming familiarity with the work [L3]. For a more general result (with all details fully worked out), see [AM3].

Given a simple map  $f$ , one can define (as in §3 of [L3]) a sequence of *holomorphic families of generalized quadratic-like maps*  $R_i$ ,  $i \geq 1$ , related by *generalized renormalization*. To fix notation, the parameter space of those families will be denoted  $\Lambda_i[f]$ , so for each  $g \in \Lambda_i[f]$  the family defines a *generalized quadratic-like map*  $R_i[g] : U_i^j[g] \rightarrow U_i[g]$ . Moreover, the family  $R_i$  is *equipped* (with a holomorphic motion  $h_i$  of the  $U_i^j$  and  $U_i$ ) and *proper*. The following properties of this sequence of families will be important for us:

- (1)  $\Lambda_i[f] \cap \mathbb{R} = J_i[f]$ , and for  $g \in J_i[f]$ ,  $U_i[g] \cap \mathbb{R} = I_i[g]$  and  $U_i^j[g] \cap \mathbb{R} = I_i^j[g]$ .
- (2) For  $g \in J_i[f]$ , the map  $R_i[g] : \cup U_i^j[g] \rightarrow U_i[g]$  is an extension of the *real first return map*  $R_i[g] : I_i^j[g] \rightarrow I_i[g]$  defined before. For  $j \neq 0$  (respectively for  $j = 0$ ),  $R_i[g] : U_i^j[g] \rightarrow U_i[g]$  is a homeomorphism (respectively a double covering).
- (3) The modulus of  $U_i[f] \setminus \overline{U_{i+1}[f]}$  grows at least linearly in  $i$  ([GS2], [L2]).
- (4) The modulus of  $\Lambda_i[f] \setminus \overline{\Lambda_{i+1}[f]}$  grows at least linearly in  $i$  ([L3]).

Define  $\tau_i$  as before by  $R_i[f](0) \in I_i^{\tau_i}[f]$ . Let  $\Lambda_i^{\tau_i}[f]$  denote the set of  $g \in \Lambda_i[f]$  such that  $R_i[g](0) \in U_i^{\tau_i}[g]$ . We have:

- (5)  $\Lambda_i^{\tau_i}[f] \cap \mathbb{R} = J_i^{\tau_i}[f]$  and in particular  $\Lambda_i^{\tau_i}[f] \cap \mathbb{R} = J_i^{\tau_i}[f]$ .

By Lemma 4.8 of [L3], the holomorphic motion  $h_i$  of  $U_i$ ,  $U_i^j$  (corresponding to  $R_i$ ) has uniformly bounded dilatation (independently of  $i$ ) *when restricted to  $U_i \setminus \overline{U_i^0}$* . Item (3) above implies that for  $i$  big, there is an annulus of big modulus (linear



growth in  $i$ ) contained in  $U_i[f] \setminus \overline{(U_i^0[f] \cup U_i^{\tau_i}[f])}$  and going around  $U_i^{\tau_i}[f]$ . By transverse quasiconformality of holomorphic motions (Corollary 2.1 of [L3]), and the  $\lambda$ -Lemma of [MSS], this estimate can be transferred to the parameter space, so we get:

(6) The modulus of  $\Lambda_i[f] \setminus \overline{\Lambda_i^{\tau_i}[f]}$  grows at least linearly in  $i$ .

For each  $g \in \Lambda_i[f]$ , denote by  $L_i[g]$  the first landing map to  $U_i^0[g]$  obtained by iteration of non-central branches of  $R_i[g]$ . By item (2) above, we have:

(7) For  $g \in J_i[f]$ , the domain of  $L_i[g]$  is a union  $\cup_{\underline{d} \in \Omega} W_i^{\underline{d}}[g]$  of disks such that  $W_i^{\underline{d}}[g] \cap \mathbb{R} = C_i^{\underline{d}}[g]$  and  $L_i[g]$  extends the *real first landing map*  $L_i[g] : \cup_{\underline{d} \in \Omega} C_i^{\underline{d}}[g] \rightarrow I_i^0[g]$  defined before.

The family  $L_i$  is also equipped with a holomorphic motion  $\hat{h}_i$  of  $U_i$  and the  $W_i^{\underline{d}}$  (see §3.5 of [L3]). Define  $\Gamma_i^{\underline{d}}[f]$  as the set of  $g \in \Lambda_i[f]$  such that  $R_i[g](0) \in W_i^{\underline{d}}[g]$ . The  $\lambda$ -Lemma and (6) imply:

(8) For  $g \in J_i^{\tau_i}[f]$ , there exists a real-symmetric qc map of  $\mathbb{C}$ , whose dilatation goes to 1 as  $i$  grows, taking  $U_i[f]$  to  $U_i[g]$ , and taking any  $W_i^{\underline{d}}[f]$  to  $W_i^{\underline{d}}[g]$ .

Items (7) and (8) prove PhPh1 in the Phase-Parameter relation.

Item (6) and transverse quasiconformality of holomorphic motions imply:

(9) There is a real-symmetric qc map of  $\mathbb{C}$ , whose dilatation goes to 1 as  $i$  grows, taking  $U_i^{\tau_i}[f]$  to  $\Lambda_i^{\tau_i}[f]$ , and taking any  $W_i^{\underline{d}}[f]$  *contained in*  $U_i^{\tau_i}[f]$  to  $\Gamma_i^{\underline{d}}[f]$ .

Items (5), (7) and (9) prove PhPa1 in the Phase-Parameter relation.

Notice that for any  $g \in \Lambda_i[f]$ , and for every  $\underline{d}$  the map  $L_i[g] : W_i^{\underline{d}}[g] \rightarrow U_i^0[g]$  extends to a holomorphic diffeomorphism  $R_i^{\underline{d}}[g] : U_i^{\underline{d}}[g] \rightarrow U_i[g]$ . It is easy to see that  $\hat{h}_i$  (as defined in §3.5 of [L3]) is also a holomorphic motion of the  $U_i^{\underline{d}}$ .

Define  $\tilde{\Lambda}_{i+1}[f]$  as the set of  $g$  such that  $R_i[g](0) \in U_i^{\underline{d}_i}[g]$ , where  $\underline{d}_i$  is chosen such that  $R_i[f](0) \in C_i^{\underline{d}_i}[f]$ . It follows that  $\Lambda_{i+1}[f] = \Gamma_i^{\underline{d}_i}[f] \subset \tilde{\Lambda}_{i+1}[f] \subset \Lambda_i^{\tau_i}[f]$ . By (3), the modulus of  $U_i^{\underline{d}_i}[f] \setminus \overline{W_i^{\underline{d}_i}[f]}$  grows at least linearly in  $i$ . By (6) and transverse quasiconformality of holomorphic motions, this implies:

(10) The modulus of  $\tilde{\Lambda}_i[f] \setminus \overline{\Lambda_i[f]}$  grows at least linearly in  $i$ .

For  $g \in \Lambda_i[f]$ , the map  $R_i[g] : U_i^0[g] \rightarrow U_i[g]$  extends to a bigger domain  $\tilde{U}_{i+1}[g] = (R_{i-1}[g]|_{U_i[g]})^{-1}(U_{i-1}^{\underline{d}_{i-1}}[g])$ , as a double covering map onto  $U_{i-1}[g]$  (notice that  $U_i^0[g] \subset \tilde{U}_{i+1}[g] \subset U_i[g]$ ). It follows:

(11) If  $g \in J_i[f]$  then  $\tilde{U}_{i+1}[g] \cap \mathbb{R} = \tilde{I}_{i+1}[g]$ .

The holomorphic motion  $\hat{h}_{i-1}$  (corresponding to  $L_{i-1}$ ) naturally lifts to a holomorphic motion  $\tilde{h}_i$  of  $U_i$ ,  $\tilde{U}_{i+1}$  and all  $U_i^j$  *not contained in*  $\tilde{U}_{i+1}$ , which is defined (in principle) over  $\Lambda_i[f]$ , but extends to a holomorphic motion defined over  $\tilde{\Lambda}_i[f]$ .

Item (10) and yet another application of the  $\lambda$ -Lemma imply:

(12) For  $g \in J_i[f]$ , there exists a real-symmetric qc map of  $\mathbb{C}$ , whose dilatation goes to 1 as  $i$  grows, taking  $U_i[f]$  to  $U_i[g]$ , and taking any  $U_i^j[f]$  *not contained in*  $\tilde{U}_{i+1}[f]$  to  $U_i^j[g]$ .

Items (2), (11) and (12) prove PhPh2 in the Phase-Parameter relation.

Item (10) and transverse quasiconformality of holomorphic motions imply:

(13) There is a real-symmetric qc map of  $\mathbb{C}$ , whose dilatation goes to 1 as  $i$  grows, taking  $U_i[f]$  to  $\Lambda_i[f]$ , and taking any  $U_i^j[f]$  *not contained in*  $\tilde{U}_{i+1}[f]$  to  $\Lambda_i^j[f]$ .

Items (2), (11) and (13) prove PhPa2 in the Phase-Parameter relation. All items of the Phase-Parameter relation are proved.

## REFERENCES

- [A] A. Avila. Bifurcations of unimodal maps: the topological and metric picture. Thesis IMPA (2001) ([www.math.sunysb.edu/~artur](http://www.math.sunysb.edu/~artur)).
- [ALM] A. Avila, M. Lyubich, W. de Melo. Regular or stochastic dynamics in real analytic families of unimodal maps. Preprint IMS at Stony Brook, #2001/15. To appear in *Inventiones Math.*
- [AM1] A. Avila, C. G. Moreira. Statistical properties of unimodal maps: smooth families with negative Schwarzian derivative. Preprint ([www.arXiv.org](http://www.arXiv.org)). To appear in *Astérisque*.
- [AM2] A. Avila, C. G. Moreira. Bifurcations of unimodal maps. Preprint ([www.arXiv.org](http://www.arXiv.org)).
- [AM3] A. Avila, C. G. Moreira. Phase-Parameter relation and sharp statistical properties in general families of unimodal maps. Preprint ([www.arXiv.org](http://www.arXiv.org)).
- [BBM] V. Baladi, M. Benedicks, V. Maume. Almost sure rates of mixing for i.i.d. unimodal maps. *Ann. Sci. Ecole Norm. Sup.* (4), v. 35 (2002), no. 1, 77-126.
- [BV] V. Baladi, M. Viana. Strong stochastic stability and rate of mixing for unimodal maps. *Ann. Sci. Ecole Norm. Sup.* (4), v. 29 (1996), no. 4, 483-517.
- [BC1] M. Benedicks, L. Carleson. On iterations of  $1 - ax^2$  on  $(-1,1)$ . *Ann. Math.*, v. 122 (1985), 1-25.
- [BC2] M. Benedicks, L. Carleson. On dynamics of the Hénon map. *Ann. Math.*, v. 133 (1991), 73-169.
- [GS1] J. Graczyk, G. Świątek. Generic hyperbolicity in the logistic family. *Ann. of Math.*, v. 146 (1997), 1-52.
- [GS2] J. Graczyk, G. Świątek. Induced expansion for quadratic polynomials. *Ann. Sci. Ecole Norm. Sup.*, IV. Sr. 29, No.4 (1996), 399-482.
- [HK] F. Hofbauer, G. Keller. Quadratic maps without asymptotic measure. *Comm. Math. Physics*, v. 127 (1990), 319-337.
- [J] M. Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Comm. Math. Phys.*, v. 81 (1981), 39-88.
- [Jo] S. D. Johnson. Singular measures without restrictive intervals. *Comm. Math. Phys.*, 110 (1987), 185-190.
- [KN] G. Keller, T. Nowicki. Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps. *Comm. Math. Phys.*, 149 (1992), 31-69.
- [L1] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. *Ann. Math.*, **140** (1994), 347-404.
- [L2] M. Lyubich. Dynamics of quadratic polynomials, I-II. *Acta Math.*, **178** (1997), 185-297.
- [L3] M. Lyubich. Dynamics of quadratic polynomials, III. Parapuzzle and SBR measure. *Astérisque*, v. 261 (2000), 173 - 200.
- [L4] M. Lyubich. Feigenbaum-Coulet-Tresser universality and Milnor's hairiness conjecture. *Ann. of Math.* (2) 149 (1999), no. 2, 319-420.
- [L5] M. Lyubich. Almost every real quadratic map is either regular or stochastic. *Ann. of Math.* (2) 156 (2002), no. 1, 1-78.
- [MN] M. Martens, T. Nowicki. Invariant measures for Lebesgue typical quadratic maps. *Astérisque*, v. 261 (2000), 239 - 252.
- [MSS] R. Mañé, P. Sad & D. Sullivan. On the dynamics of rational maps, *Ann. Sci. Ecole Norm. Sup.*, **16** (1983), 193-217.
- [MvS] W. de Melo, S. van Strien. *One-dimensional dynamics*. Springer, 1993.
- [NS] T. Nowicki, D. Sands. Non-uniform hyperbolicity and universal bounds for  $S$ -unimodal maps. *Invent. Math.* 132 (1998), no. 3, 633-680.
- [Pa] J. Palis. A global view of dynamics and a conjecture of the denseness of finitude of attractors. *Astérisque*, v. 261 (2000), 335 - 347.
- [Y] L.-S. Young. Decay of correlations for certain quadratic maps. *Comm. Math. Phys.*, 146 (1992), 123-138.

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